An Introduction to Black Holes
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Part I

Black Holes

Those notes are meant to accompany my lectures at the CIMPA-UNESCO-MOROCCO School "Riemannian Geometry, pseudo-Riemannian Geometry and Mathematical Physics", May 19-30, 2008, Faculty of Science and Techniques, Marrakech, Morocco. These are *not* lecture notes, as I will only lecture on a very small part of the material covered here: I am planning a detailed introduction to the Schwarzschild metric, followed by a concise overview of the Kerr and of the Emparan-Reall ones. I plan to finish with an outline of the proof of uniqueness of static black holes. I hope that the notes will be useful to the participants in further studies of the subject.

I will *not* lecture on the material contained in the appendix at all; it is hoped that the participants of the school will already be familiar with a substantial part of the material presented there and so this is included here for self-study *before the lectures* for those who are not, but also to fix conventions and notations.

The readers of those notes are invited to inform me of any misprints and/or mistakes, please send an email to `chrusciel@maths.ox.ac.uk`

Chapter 1

Fundamentals

Black holes belong to the most fascinating objects predicted by Einstein's theory of gravitation. Although they have been studied for years,[1] they still attract tremendous attention in the physics and astrophysics literature. It turns out that several field theories are known to possess solutions which exhibit black hole properties:

- The "standard" gravitational ones which, according to our current postulates, are black holes for all classical fields.
- The "dumb holes", which are the sonic counterparts of black holes, first discussed by Unruh [72].
- The "optical" ones – the black-hole counterparts arising in the theory of moving dielectric media, or in non-linear electrodynamics [37, 55].
- The "numerical black holes" – objects constructed by numerical general relativists.

(An even longer list of models and submodels can be found in [1].) In this work we shall discuss various aspects of the above. The reader is referred to [3, 17, 25, 28, 61, 73] and references therein for a review of quantum aspects of black holes. We start with a short review of the observational status of black holes in astrophysics.

1.1 Black holes as astrophysical objects

When a star runs out of nuclear fuel, it must find ways to fight gravity. Current physics predicts that dead stars with masses up to the Chandrasekhar limit, $M_{\mathrm{mcH}} = 1.4M_\odot$, become white dwarfs, where electron degeneracy supplies the necessary pressure. Above the Chandrasekhar limit, and up to a second mass limit, $M_{\mathrm{NS,max}} \sim 2 - 3M_\odot$, dead stars are expected to become neutron stars, where neutron degeneracy pressure holds them up. If a dead star has a mass

[1] The reader is referred to the introduction to [9] for an excellent concise review of the history of the concept of a black hole, and to [8, 27] for more detailed ones.

$M > M_{\text{NS,max}}$, there is no known force that can hold the star up. What we have then is a black hole.

While there is growing evidence that black holes do indeed exist in astrophysical objects, and that alternative explanations for the observations discussed below seem less convincing, it should be borne in mind that no undisputed evidence of occurrence of black holes has been presented so far. The flagship black hole candidate used to be Cygnus X-1, known and studied for years (*cf., e.g.,* [9]), and it still remains a strong one. Table 1.1[2] lists a series of further strong black hole candidates in X-ray binary systems; M_c is mass of the compact object and M_* is that of its optical companion; some other candidates, as well as references, can be found in [45, 53]; a very readable overview of the observations can be found in [50]. The binaries have been divided into two families: the High Mass X-ray Binaries (HMXB), where the companion star is of (relatively) high mass, and the Low Mass X-ray Binaries (LMXB), where the companion is typically below a solar mass. The LMXB's include the "X-ray transients", so-called because of flaring-up behaviour. This particularity allows to make detailed studies of their optical properties during the quiescent periods, which would be impossible during the periods of intense X-ray activity. The stellar systems listed have X-ray spectra which are neither periodic (that would correspond to a rotating neutron star), nor recurrent (which is interpreted as thermonuclear explosions on a neutron star's hard surface). The final selection criterion is that of the mass M_c exceeding the Chandrasekhar limit $M_C \approx 3$ solar masses $M_\odot$.[3] According to the authors of [9], the strongest black hole candidate in 1999 was V404 Cygni, which belongs to the LMXB class. Table 1.1 should be put into perspective by realizing that, by some estimates [39], a typical galaxy – such as ours – should harbour $10^7 - 10^8$ stellar black mass holes. We note an interesting proposal, put forward in [10], to carry out observations by gravitational microlensing of some 20 000 stellar mass black holes that are predicted [47] to cluster within 0.7 pc of Sgr A* (the centre of our galaxy).

[2]The recent review [53] lists thirteen black hole candidates.

[3]See [45] for a discussion and references concerning the value of M_C.

Table 1.1: Stellar mass black hole candidates (from [39])

Type	Binary system	$M_c/M_\odot$	$M_*/M_\odot$
HMXB:	Cygnus X-1	11–21	24–42
	LMC X-3	5.6 –7.8	20
	LMC X-1	≥ 4	4–8
LMXB:	V 404 Cyg	10–15	≈ 0.6
	A 0620-00	5–17	0.2–0.7
	GS 1124-68 (Nova Musc)	4.2–6.5	0.5–0.8
	GS 2000+25 (Nova Vul 88)	6-14	≈ 0.7
	GRO J 1655-40	$4.5 - 6.5$	≈ 1.2
	H 1705-25 (Nova Oph 77)	5–9	≈ 0.4
	J 04224+32	6–14	$\approx 0.3 - 0.6$

Table 1.2: Twenty-nine supermassive black hole candidates (from [32, 46])

dynamics of	host galaxy	$M_h/M_\odot$	host galaxy	$M_h/M_\odot$
water maser discs:	NGC 4258	4×10^7		
gas discs:	IC 1459	2×10^8	M 87	3×10^9
	NGC 2787	4×10^7	NGC 3245	2×10^8
	NGC 4261	5×10^8	NGC 4374	4×10^8
	NGC 5128	2×10^8	NGC 6251	6×10^8
	NGC 7052	3×10^8		
stars:	NGC 821	4×10^7	NGC 1023	4×10^7
	NGC 2778	1×10^7	NGC 3115	1×10^9
	NGC 3377	1×10^8	NGC 3379	1×10^8
	NGC 3384	1×10^7	NGC 3608	1×10^8
	NGC 4291	2×10^8	NGC 4342	3×10^8
	NGC 4473	1×10^8	NGC 4486B	5×10^8
	NGC 4564	6×10^7	NGC 4649	2×10^9
	NGC 4697	2×10^8	NGC 4742	1×10^7
	NGC 5845	3×10^8	NGC 7457	4×10^6
	Milky Way	2.5×10^6		

It is now widely accepted that quasars and active galactic nuclei are powered by accretion onto massive black holes [41, 76]. Further, over the last few years there has been increasing evidence that massive dark objects may reside at the centres of most, if not all, galaxies [40, 64]. In several cases the best explanation for the nature of those objects is that they are "heavyweight" black holes, with masses ranging from 10^6 to 10^{10} solar masses. Table 1.2[4] lists some supermassive black hole candidates; some other candidates, as well as precise references, can be found in [32, 45, 46, 63]. The main criterion for finding candidates for such black holes is the presence of a large mass within a small region; this is determined by maser line spectroscopy, gas spectroscopy, or by measuring the motion of stars orbiting around the galactic nucleus. The reader is referred to [49] for a discussion of the maser emission lines and their analysis for the supermassive black hole candidate NGC 4258. An example of measurements via gas spectrography is given by the analysis of the Hubble Space Telescope (HST)

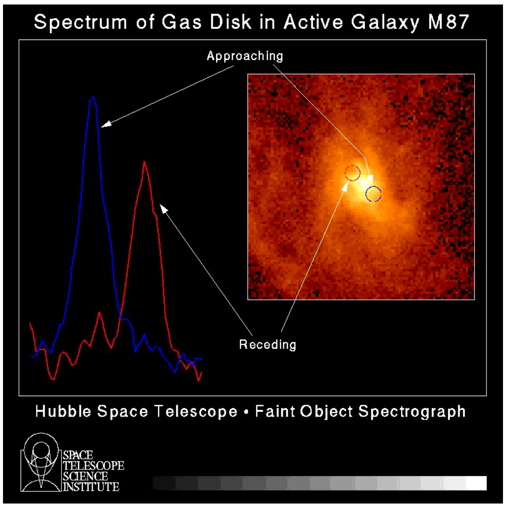

Figure 1.1: Hubble Space Telescope observations of spectra of gas in the vicinity of the nucleus of the radio galaxy M 87, NASA and H. Ford (STScI/JHU) [67].

observations of the radio galaxy M 87 [71] (compare [41]): A spectral analysis shows the presence of a disk-like structure of ionized gas in the innermost few arc seconds in the vicinity of the nucleus of M 87. The velocity of the gas measured by spectroscopy (cf. Fig. 1.1) at a distance from the nucleus of the order of 6×10^{17} m, shows that the gas recedes from us on one side, and approaches us on the other, with a velocity difference of about 920 km s^{-1} . This leads to a

[4] The table lists those galaxies which are listed both in [46] and [32]; we note that some candidates from earlier lists [63] do not occur any more in [32, 46]. Nineteen of the observations listed have been published in 2000 or 2001.

mass of the central object of $\sim 3 \times 10^9\ M_\odot$, and no form of matter can occupy such a small region except for a black hole. Figure 1.2 shows another image, reconstructed out of HST observations, of a recent candidate for a supermassive black hole – the (active) galactic nucleus of NGC 4438 [30].

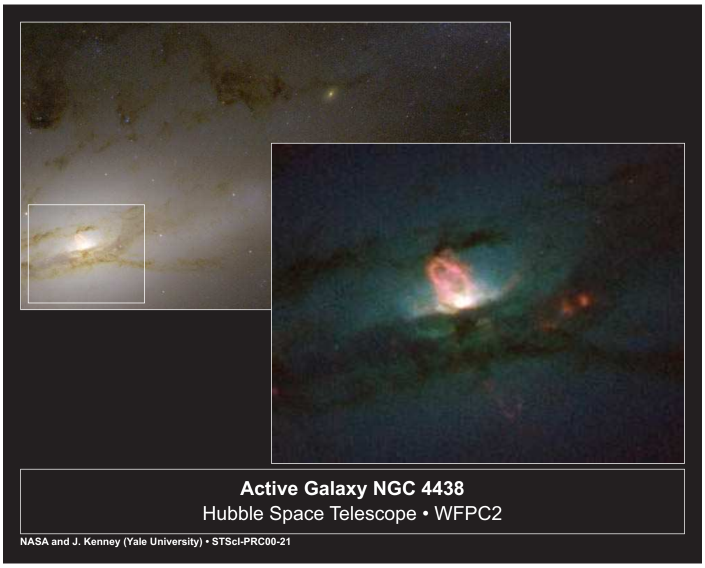

Figure 1.2: Hubble Space Telescope observations [30] of the nucleus of the galaxy NGC 4438, from the STScI Public Archive [67].

To close the discussion of Table 1.2, we note that the determination of mass of the galactic nuclei via direct measurements of star motions has been made possible by the unprecedentedly high angular resolution and sensitivity of the HST, see also Figure 1.3.

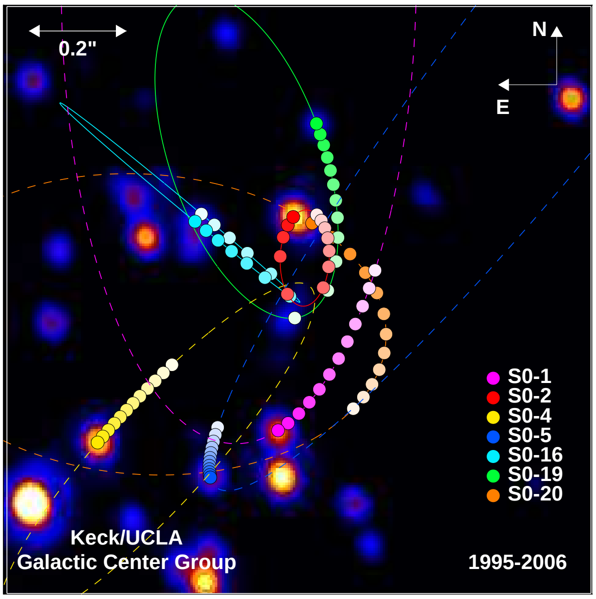

Figure 1.3: The orbits of stars within the central 1.0 × 1.0 arcseconds of our Galaxy. In the background, the central portion of a diffraction-limited image taken in 2006 is displayed. While every star in this image has been seen to move over the past 12 years, estimates of orbital parameters are only possible for the seven stars that have had significant curvature detected. The annual average positions for these seven stars are plotted as colored dots, which have increasing color saturation with time. Also plotted are the best fitting simultaneous orbital solutions. These orbits provide the best evidence yet for a supermassive black hole, which has a mass of 3.7 million times the mass of the Sun. The image was created by Andrea Ghez and her research team at UCLA, from data sets obtained with the W. M. Keck Telescopes, and is available at `http://www.astro.ucla.edu/~ghezgroup/gc/pictures/`.

There seems to be consensus [32, 46, 64] that the two most convincing supermassive black hole candidates are the galactic nuclei of NGC 4258 and of our own Milky Way.

There have been suggestions for existence for an intermediate-mass black hole orbiting three light-years from Sagittarius A*. This black hole of 1,300 solar masses is within a cluster of seven stars, possibly the remnant of a massive star cluster that has been stripped down by the Galactic Centre [42].

A compilation of a list of black hole candidates, some very tentative, can be found at `http://www.johnstonsarchive.net/relativity/bhctable.html`, see also [77].

Let us close this section by pointing out the review paper [6] which discusses both theoretical and experimental issues concerning *primordial* black holes.

1.2 The Schwarzschild solution and its extensions

Stationary solutions are of interest for a variety of reasons. As models for compact objects at rest, or in steady rotation, they play a key role in astrophysics. They are easier to study than non-stationary systems because stationary solutions are governed by elliptic rather than hyperbolic equations. Further, like in any field theory, one expects that large classes of dynamical solutions approach a stationary state in the final stages of their evolution. Last but not least, explicit stationary solutions are easier to come by than dynamical ones.

The simplest stationary solutions describing compact isolated objects are the spherically symmetric ones. A theorem due to Birkhoff shows that in the vacuum region any spherically symmetric metric, even without assuming stationarity, belongs to the family of Schwarzschild metrics, parameterized by a mass parameter m:

$$g = -V^2dt^2 + V^{-2}dr^2 + r^2d\Omega^2 \ , \tag{1.2.1}$$

$$V^2 = 1 - \tfrac{2m}{r} \ , \quad t \in \mathbb{R} \ , \ r \in (2m, \infty) \ . \tag{1.2.2}$$

Here $d\Omega^2$ denotes the metric of the standard 2-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \ .$$

Throughout this section we will assume

$$m > 0 \ ,$$

because $m < 0$ leads to metrics which are "nakedly singular", in the following sense: for $m < 0$, on each space-like surface $\{t = \text{const}\}$ the set $\{r = 0\}$ can be reached along curves of finite length. But we have (see, e.g., `http://grtensor.phy.queensu.ca/NewDemo`)

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} \ , \tag{1.2.3}$$

which shows that the geometry is singular there.

One of the first features one notices is that the metric (1.2.1) is singular as $r = 2m$ is approached. It turns out that this singularity is related to a poor choice of coordinates (one talks about "a coordinate singularity"); the simplest way to see it is to replace t by a new coordinate v defined as

$$v = t + f(r) \ , \quad f' = \frac{1}{V^2} \ . \tag{1.2.4}$$

Now,

$$f' = \frac{1}{1 - \frac{2m}{r}} = \frac{r}{r - 2m} = \frac{r - 2m + 2m}{r - 2m} = 1 + \frac{2m}{r - 2m} \ , \tag{1.2.5}$$

leading to

$$v = t + r + 2m\ln(r - 2m) .$$

This brings g to the form

$$g = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2 d\Omega^2 . \qquad (1.2.6)$$

We have $\det g = -r^4 \sin^2\theta$, with all coefficients of g smooth, which shows that g is a well defined Lorentzian metric on the set

$$v \in \mathbb{R} , \quad r \in (0, \infty) . \qquad (1.2.7)$$

More precisely, (1.2.6)-(1.2.7) is an analytic extension of the original space-time (1.2.1).

It is easily seen that the region $\{r \le 2m\}$ for the metric (1.2.6) is a *black hole region*, in the sense that

observers, or signals, can enter this region, but can never leave it. (1.2.8)

In order to see that, recall that observers in general relativity always move on *future directed timelike curves*, that is, curves with timelike future directed tangent vector. For signals the curves are *causal future directed*, these are curves with timelike or null future directed tangent vector. Let, then, $\gamma(s) = (v(s), r(s), \theta(s), \varphi(s))$ be such a timelike curve, for the metric (1.2.6) the timelikeness condition $g(\dot\gamma, \dot\gamma) < 0$ reads

$$-(1 - \frac{2m}{r})\dot v^2 + 2\dot v \dot r + r^2(\dot\theta^2 + \sin^2\theta \dot\varphi^2) < 0 .$$

This implies

$$\dot v\Big(- (1 - \frac{2m}{r})\dot v + 2\dot r\Big) < 0 .$$

It follows that $\dot v$ does not change sign on a timelike curve. The usual choice of time orientation corresponds to $\dot v > 0$ on future directed curves (compare (1.2.4)), leading to

$$-(1 - \frac{2m}{r})\dot v + 2\dot r < 0 .$$

For $r \le 2m$ the first term is non-negative, which enforces $\dot r < 0$ on all future directed timelike curves in that region. Thus, r is a strictly decreasing function along such curves, which implies that future directed timelike curves can cross the hypersurface $\{r = 2m\}$ only if coming from the region $\{r > 2m\}$. This motivates the name *black hole event horizon* for $\{r = 2m, v \in \mathbb{R}\}$. The same conclusion applies for causal curves: it suffices to approximate a causal curve by a sequence of timelike ones.

Note that we could have chosen a time orientation in which future directed timelike curves satisfy $\dot v < 0$. The resulting space-time is then called a *white hole* space-time, with $\{r = 2m\}$ being a *white hole event horizon*, which can only be crossed by those future directed causal curves which originate in the region $\{r < 2m\}$.

From (1.2.6) one easily finds the inverse metric:

$$g^{\mu\nu}\partial_\mu\partial_\nu = 2\partial_v\partial_r + (1-\frac{2m}{r})\partial_r^2 + r^{-2}\partial_\theta^2 + r^{-2}\sin^{-2}\theta\partial_\varphi^2 \ . \tag{1.2.9}$$

In particular

$$0 = g^{vv} = g(\nabla v, \nabla v) \ ,$$

which implies that the integral curves of

$$\nabla v = \partial_r$$

are null, affinely parameterised geodesics. This is at the origin of the coordinate system (v, r, θ, φ).

We also have

$$g(\nabla r, \nabla r) = g^{rr} = 1 - \frac{2m}{r} \ , \tag{1.2.10}$$

so that the the surface $r = 2m$ is *null*. It is reached by all the radial null geodesics $v = \text{const}$, $\theta = \text{const}'$, $\varphi = \text{const}''$ in finite affine time.

An alternative justification of the property, that the horizon $\{r = 2m\}$ cannot be reached by future directed timelike curves starting in the region $\{r < 2m\}$, is provided by (1.2.10), which shows that r is a *time function* in $\{r < 2m\}$.

1.2.1 The Kruskal-Szekeres extension

The transition from (1.2.1) to (1.2.6) is not the end of the story, as further extensions are possible, which will be clear from the calculations that we will do shortly. For the metric (1.2.1) a maximal analytic extension has been found independently by Kruskal [33], Szekeres [69], and Fronsdal [20]; for some obscure reason Fronsdal is almost never mentioned in this context. This extension is visualised[5] in Figure 1.4. The region I there corresponds to the space-time (1.2.1), while the extension just constructed corresponds to the regions I and II.

The general construction, for spherically symmetric metrics, proceeds as follows: we introduce another coordinate u defined by changing a sign in (1.2.4)

$$u = t - f(r) \ , \quad f' = \frac{1}{V^2} \ , \tag{1.2.11}$$

leading to

$$u = t - r - 2m\ln(r - 2m) \ .$$

We could now replace (t, r) by (u, r), obtaining an extension of the exterior region I of Figure 1.4 into the "white hole" region IV. We leave that extension as an exercise for the reader, and we pass to the complete extension, which proceeds in two steps. First, we replace (t, r) by (u, v). We note that

$$V\,du = V\,dt - \frac{1}{V}dr \ , \quad V\,dv = V\,dt + \frac{1}{V}dr \ ,$$

[5] I am grateful to J.-P. Nicolas for allowing me to use his electronic figures [54].

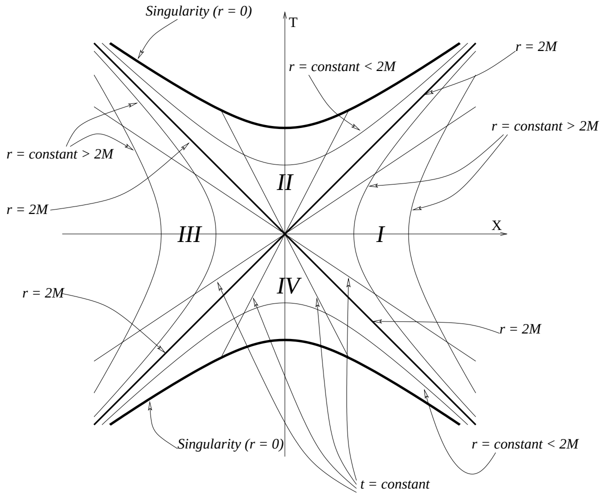

Figure 1.4: The Kruskal-Szekeres extension of the Schwarzschild solution.

which gives

$$V\,dt = \frac{V}{2}(du+dv)\ ,\quad \frac{1}{V}dr = \frac{V}{2}(dv-du)\ .$$

Inserting this into (1.2.1) brings g to the form

$$\begin{aligned} g &= -V^2dt^2 + V^{-2}dr^2 + r^2d\Omega^2 \\ &= \frac{V^2}{4}\Big(-(du+dv)^2 + (du-dv)^2\Big) + r^2d\Omega^2 \\ &= -V^2du\,dv + r^2d\Omega^2\ . \end{aligned} \tag{1.2.12}$$

The metric so obtained is still degenerate at $\{V=0\}$. The desingularisation is now obtained by setting

$$\hat u = -\exp(-cu)\ ,\quad \hat v = \exp(cv)\ , \tag{1.2.13}$$

with an appropriately chosen c: since

$$d\hat u = c\exp(-cu)\,du\ ,\quad d\hat v = c\exp(cv)dv\ ,$$

we obtain

$$\begin{aligned} V^2du\,dv &= \frac{V^2}{c^2}\exp(-c(-u+v))d\hat u\,d\hat v \\ &= \frac{V^2}{c^2}\exp(-2cf(r))d\hat u\,d\hat v\ . \end{aligned}$$

In the Schwarzschild case this reads

$$\begin{aligned} \frac{V^2}{c^2}\exp(2cf(r)) &= \frac{r-2m}{c^2r}\exp(-2c(r+2m\ln(r-2m))) \\ &= \frac{\exp(-2cr)}{c^2r}(r-2m)\exp(-4mc\ln(r-2m))\ , \end{aligned}$$

and with the choice

$$4mc = 1$$

the term $r-2m$ cancels out, leading to a factor in front of $d\hat{u}\,d\hat{v}$ which has no zeros for $r \neq 0$ near. Thus, the desired coordinate transformation is

$$\hat{u} = -\exp(-cu) = -\exp(\tfrac{r-t}{4m})\sqrt{r-2m}\ , \tag{1.2.14}$$
$$\hat{v} = \exp(cv) = \exp(\tfrac{r+t}{4m})\sqrt{r-2m}\ , \tag{1.2.15}$$

with g taking the form

$$\begin{aligned} g &= -V^2 du\,dv + r^2 d\Omega^2 \\ &= -\frac{16m^2\exp(-\frac{r}{2m})}{r} d\hat{u}\,d\hat{v} + r^2 d\Omega^2\ . \end{aligned} \tag{1.2.16}$$

Here r should be viewed as a function of $\hat{u}$ and $\hat{v}$ defined implicitly by the equation

$$-\hat{u}\hat{v} = \underbrace{\exp(\frac{r}{2m})(r-2m)}_{=:G(r)}\ . \tag{1.2.17}$$

Indeed, we have

$$\Big(\exp(\frac{r}{2m})(r-2m)\Big)' = \frac{r}{2m}\exp(\frac{r}{2m}) > 0\ ,$$

which shows that the function G defined at the right-hand-side of (1.2.17) is a smooth strictly increasing function of $r>0$. We have $G(0) = -2m$, and G tends to infinity as r does, so G defines a bijection of $(0,\infty)$ with $(-2m,\infty)$. The implicit function theorem guarantees smoothness of the inverse G^{-1}, and hence the existence of a smooth function $r = G^{-1}(-\hat{u}\hat{v})$ solving (1.2.17) on the set $\hat{u}\hat{v} \in (-\infty, 2m)$.

We have $\det g = -\frac{\exp(-\frac{r}{m})}{(16)^2 m^4} r^2\sin^2\theta$, with all coefficients of g smooth, which shows that (1.2.16) defines a smooth Lorentzian metric on the set

$$\hat{u},\ \hat{v} \in \mathbb{R}\ , \quad r > 0\ . \tag{1.2.18}$$

This is the *Kruszkal-Szekeres* extension of the original space-time (1.2.1). Figure 1.4 gives a representation of the extended space-time in coordinates

$$X = (\hat{v}-\hat{u})/2\ , \quad T = (\hat{v}+\hat{u})/2\ .$$

Since (1.2.3) shows that the so-called *Kretschmann scalar* $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges as r^{-6} when r approaches zero, we conclude that the metric cannot be extended across the set $r=0$, at least in the class of C^2 metrics.

Let us discuss some features of Figure 1.4:

1. The singular set $r=0$ corresponds to the spacelike hyperboloids

$$(X^2 - T^2)|_{r=0} = -\hat{u}\hat{v}|_{r=0} = 2m > 0\ .$$

2. More generally, the sets $r = \text{const}$ are hyperboloids $X^2 - T^2 = \text{const}'$, which are timelike in the regions I and III (since $X^2 - T^2 < 0$ there), and which are spacelike in the regions II and IV.

3. The vector field ∇T satisfies

$$g(\nabla T, \nabla T) = g^\sharp(dT, dT) = \frac{1}{4}g^\sharp(d\hat{u} + d\hat{v}, d\hat{u} + d\hat{v}) = \frac{1}{2}g^\sharp(d\hat{u}, d\hat{v}) < 0 \ ,$$

which shows that T is a time coordinate. Similarly X is a space-coordinate, so that Figure 1.4 respects our implicit convention of representing time along the vertical axis and space along the horizontal one.

4. The map

$$(\hat{u}, \hat{v}) \to (-\hat{u}, -\hat{v})$$

is clearly an isometry, so that the region I is isometric to region III, and region II is isometric to region IV. In particular the extended manifold has two asymptotically flat regions, the original region I, and region III which is an identical copy I.

5. The hypersurface $t = 0$ from the region I corresponds to $\hat{u} = -\hat{v} > 0$, equivalently it is the subset $X > 0$ of the hypersurface $T = 0$. This can be smoothly continued to negative X, which corresponds to a second copy of this hypersurface. The resulting geometry is often referred to as the *Einstein-Rosen bridge*. It is instructive to do the continuation directly using the Riemannian metric γ induced by g on $t = 0$:

$$\gamma = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 \ , \quad r > 2m \ .$$

A convenient coordinate ρ is given by

$$\rho = \sqrt{r^2 - 4m^2} \iff r = \sqrt{\rho^2 + 4m^2} \ .$$

This brings γ to the form

$$\gamma = \Big(1 + \frac{2m}{\sqrt{\rho^2 + 4m^2}}\Big) d\rho^2 + (\rho^2 + 4m^2) d\Omega^2 \ , \tag{1.2.19}$$

which can be smoothly continued from the original range $\rho > 0$ to $\rho \in \mathbb{R}$. Equation (1.2.19) further exhibits explicitly asymptotic flatness of both asymptotic regions $\rho \to \infty$ and $\rho \to -\infty$. Indeed,

$$g \sim d\rho^2 + \rho^2 d\Omega^2$$

to leading order, for large $|\rho|$, which is the flat metric in radial coordinates with radius $|\rho|$.

6. In the Kruskal-Szekeres coordinate system the Killing vector field $K = \partial_t$ takes the form

$$\begin{aligned} K &= \partial_t = \frac{\partial \hat{u}}{\partial t}\partial_{\hat{u}} + \frac{\partial \hat{v}}{\partial t}\partial_{\hat{v}} \\ &= -\hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}} \ . \end{aligned} \tag{1.2.20}$$

More precisely, the Killing vector field ∂_t defined on the original Schwarzschild region extends to a Killing vector field X defined throughout the Kruskal-Szekeres manifold by the right-hand-side of (1.2.20).

We note that K is tangent to the level sets of $\hat{u}$ or $\hat{v}$ at $\hat{u}\hat{v} = 0$, and therefore is null there. Moreover, it vanishes at the sphere $\hat{u} = \hat{v} = 0$, which is called *the bifurcation surface of a bifurcate Killing horizon*. The justification of this last terminology should be clear from Figure 1.4. Quite generally, a null hypersurface to which a Killing vector is tangent, and null there, is called a *Killing horizon*. Therefore the union $\{\hat{u}\hat{v} = 0\}$ of the black hole horizon $\{\hat{u} = 0\}$ and the white hole event horizon $\{\hat{v} = 0\}$ can be written as the union of four Killing horizons and of their bifurcation surface.

The bifurcate horizon structure, as well as the formula (1.2.20), are rather reminiscent of what happens when considering the Killing vector $t\partial_x + x\partial_t$ in Minkowski space-time; this is left as an exercice to the reader.

The Kruskal-Szekeres extension is *inextendible*, which can be proved as follows: first, (1.2.3) shows that the *Kretschmann scalar* $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges as r approaches zero. This implies that no C^2 extension of the metric is possible across the set $\{r = 0\}$. Next, an analysis of the geodesics of the Kruskal-Szekeres metric shows that all (maximally extended) geodesics which *do not* approach $\{r = 0\}$ are complete. This implies inextendibility.

Nevertheless, it should be realised that the exterior Schwarzschild space-time (1.2.1) admits an *infinite* number of non-isometric vacuum extensions, *even in the class of maximal, analytic, simply connected ones*: indeed, let S be any two-dimensional closed submanifold entirely included in, say, the black-hole region of the Kruskal-Szekeres manifold $(\mathscr{M}, g)$, such that $\mathscr{M} \setminus S$ is *not* simply connected. (An example is given by the sphere $\{\hat{u} = \hat{v} = 0\}$.) Then, for any such S the universal covering manifold $(\mathscr{M}_S, \hat{g})$ of $(\mathscr{M} \setminus S, g|_{\mathscr{M}\setminus S})$ has the claimed properties.

It follows from what has been said that the Kruskal-Szekeres extension is singled out by being maximal in the vacuum, analytic, simply connected class, with all maximally extended geodesics γ either complete, or with the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverging along γ in finite affine time.

1.2.2 Other coordinate systems, higher dimensions

A convenient coordinate system for the Schwarzschild metric is given by the so-called *isotropic coordinates*: introducing a new radial coordinate $\tilde{r}$, implicitly defined by the formula

$$r = \tilde{r}\Big(1 + \frac{m}{2\tilde{r}}\Big)^2 , \tag{1.2.21}$$

with a little work one obtains

$$g_m = \left(1 + \frac{m}{2|x|}\right)^4 \left(\sum_{1=1}^{3} (dx^i)^2\right) - \left(\frac{1 - m/2|x|}{1 + m/2|x|}\right)^2 dt^2 , \tag{1.2.22}$$

where x^i are coordinates on $\mathbb{R}^3$ with $|x| = \tilde{r}$. Those coordinates show explicitly that the space-part of the metric is conformally flat (as follows from spherical symmetry).

The Schwarzschild space-time has the curious property of possessing *flat* spacelike hypersurfaces. They appear miraculously when introducing the *Painlevé–Gullstrand coordinates* [22, 36, 57]: Starting from the standard coordinate system of (1.2.1) one introduces a new time τ via the equation

$$t = \tau - 2r\sqrt{\frac{2m}{r}} + 4m \ \operatorname{arctanh}\left(\sqrt{\frac{2m}{r}}\right) , \tag{1.2.23}$$

so that

$$dt = d\tau - \frac{\sqrt{2m/r}}{1-2m/r}\, dr .$$

This leads to

$$g = -\left[1-\frac{2m}{r}\right] d\tau^2 + 2\sqrt{\frac{2m}{r}}\, dr\, d\tau + dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right] ,$$

or, passing from spherical to standard coordinates,

$$g = -\left[1-\frac{2m}{r}\right] d\tau^2 + 2\sqrt{\frac{2m}{r}}\, dr\, d\tau + dx^2 + dy^2 + dz^2 . \tag{1.2.24}$$

(Note that each such slice has zero ADM mass.)

An important tool for the PDE analysis of space-times is provided by *wave coordinates*. In spherical coordinates associated to wave coordinates $(t, \hat{x}, \hat{y}, \hat{z})$, with radius function $\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2 + \hat{z}^2}$, the Schwarzschild metric takes the form [38, 68]

$$g = -\frac{\hat{r}-m}{\hat{r}+m} dt^2 + \frac{\hat{r}+m}{\hat{r}-m} d\hat{r}^2 + (\hat{r}+m)^2 d\Omega^2 . \tag{1.2.25}$$

This is clearly obtained by replacing r with $\hat{r} = r - m$ in (1.2.1).

In order to verify the harmonic character of the coordinates associated with (1.2.25), consider a general spherically symmetric static metric of the form

$$\begin{aligned} g &= -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + e^{2\gamma}r^2 d\Omega^2 \\ &= -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + e^{2\gamma}(\delta_{ij}dx^i dx^j - dr^2) \\ &= -e^{2\alpha}dt^2 + \Big(e^{2\gamma}\delta_{ij} + (e^{2\beta} - e^{2\gamma})\frac{x^i x^j}{r^2}\Big)dx^i dx^j , \end{aligned} \tag{1.2.26}$$

where α, β and γ depend only upon r. We need to calculate

$$\Box_g x^\alpha = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}g^{\mu\nu}\partial_\nu x^\alpha) = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}g^{\mu\alpha}) .$$

Clearly $g^{0i} = 0$, which makes the calculation for $x^0 = t$ straightforward:

$$\Box_g t = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}g^{\mu 0}) = \frac{1}{\sqrt{|\det g|}}\partial_t(\sqrt{|\det g|}g^{00}) = 0 ,$$

as nothing depends upon t. For $\Box_g x^i$ we have to calculate $\sqrt{|\det g|}$ and $g^{\mu\nu}$. For the latter, it is clear that $g^{00} = -e^{-2\alpha}$, while by symmetry considerations we must have

$$g^{ij} = e^{-2\gamma}\Big(\delta^{ij} + \chi\frac{x^i x^j}{r^2}\Big) ,$$

for a function χ to be determined. The equation

$$\begin{aligned}
\delta^j_i &= g^{j\mu}g_{\mu i} = g^{jk}g_{ki} = e^{-2\gamma}\Big(\delta^{jk} + \chi\frac{x^j x^k}{r^2}\Big)\Big(e^{2\gamma}\delta_{ki} + (e^{2\beta} - e^{2\gamma})\frac{x^k x^i}{r^2}\Big) \\
&= \delta^j_i + e^{-2\gamma}\Big(\chi e^{2\gamma} + e^{2\beta} - e^{2\gamma} + \chi(e^{2\beta} - e^{2\gamma})\Big)\frac{x^i x^j}{r^2} \\
&= \delta^j_i + e^{-2\gamma}\Big(e^{2\beta} - e^{2\gamma} + \chi e^{2\beta}\Big)\frac{x^i x^j}{r^2}
\end{aligned}$$

gives $\chi = e^{2(\gamma-\beta)} - 1$, and finally

$$g^{ij} = e^{-2\gamma}\delta^{ij} + (e^{-2\beta} - e^{-2\gamma})\frac{x^i x^j}{r^2} .$$

Next, $\sqrt{|\det g|}$ is best calculated in a coordinate system in which the vector (x, y, z) is aligned along the x axis, $(x, y, z) = (r, 0, 0)$. Then (1.2.26) reads, in space-time dimension $n + 1$,

$$g = \begin{pmatrix} -e^{2\alpha} & 0 & 0 & \cdots & 0 \\ 0 & e^{2\beta} & 0 & \cdots & 0 \\ 0 & 0 & e^{2\gamma} & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & e^{2\gamma} \end{pmatrix}$$

which implies

$$\det g = -e^{2(\alpha+\beta)+2(n-1)\gamma} ,$$

still at $(x, y, z) = (r, 0, 0)$. Spherical symmetry implies that this equality holds everywhere.

In order to continue, it is convenient to set

$$\phi = e^{\alpha+\beta+(n-3)\gamma} \qquad \psi = e^{\alpha+\beta+(n-1)\gamma}(e^{-2\beta} - e^{-2\gamma}) .$$

We then have

$$\begin{aligned}
\sqrt{|\det g|}\Box_g x^i &= \partial_\mu(\sqrt{|\det g|}g^{\mu i}) = \partial_j(\sqrt{|\det g|}g^{ji}) \\
&= \partial_j\Big(\underbrace{e^{\alpha+\beta+(n-3)\gamma}}_{\phi}\delta^{ij} + \underbrace{e^{\alpha+\beta+(n-1)\gamma}(e^{-2\beta} - e^{-2\gamma})}_{\psi}\frac{x^i x^j}{r^2})\Big) \\
&= (\phi' + \psi')\frac{x^i}{r} + \psi\partial_j\Big(\frac{x^i x^j}{r^2}\Big) = \Big(\phi' + \psi' + \frac{(n-1)}{r}\psi\Big)\frac{x^i}{r} .
\end{aligned} \tag{1.2.27}$$

For the metric (1.2.25) we have

$$e^{2\alpha} = \frac{\hat r - m}{\hat r + m} , \quad \beta = -\alpha , \quad e^{2\gamma}\hat r^2 = (\hat r + m)^2 ,$$

so that

$$\phi = 1 , \quad \psi = e^{2\gamma} \times e^{2\alpha} - 1 = \frac{(\hat r + m)^2}{\hat r^2} \times \frac{\hat r - m}{\hat r + m} - 1 = -\frac{m^2}{\hat r^2} ,$$

and if $n=3$ we obtain $\Box_g x^\mu = 0$, as desired.

More generally, consider the Schwarzschild metric in any dimension $n\ge 3$,

$$g_m = -\left(1-\frac{2m}{r^{n-2}}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r^{n-2}}} + r^2 d\Omega^2\ , \qquad (1.2.28)$$

where, as usual, $d\Omega^2$ is the round unit metric on S^{n-1}. In order to avoid confusion we keep the symbol r for the coordinate appearing in (1.2.28), and rewrite (1.2.26) as

$$g \quad = \quad -e^{2\alpha}dt^2 + e^{2\beta}d\hat r^2 + e^{2\gamma}\hat r^2 d\Omega^2\ , \qquad (1.2.29)$$

It follows from (1.2.27) that the harmonicity condition reads

$$0 = \frac{d(\phi+\psi)}{d\hat r} + \frac{(n-1)}{\hat r}\psi = \frac{d(\phi+\psi)}{d\hat r} + \frac{(n-1)}{\hat r}(\psi+\phi) - \frac{(n-1)}{\hat r}\phi\ . \qquad (1.2.30)$$

Equivalently,

$$\frac{d[\hat r^{n-1}(\phi+\psi)]}{d\hat r} = (n-1)\hat r^{n-2}\phi\ . \qquad (1.2.31)$$

Transforming r to $\hat r$ in (1.2.54) and comparing with (1.2.29) we find

$$e^\alpha = \sqrt{1-\frac{2m}{r^{n-2}}}\ ,\quad e^\beta = e^{-\alpha}\frac{dr}{d\hat r}\ ,\quad e^\gamma = \frac{r}{\hat r}\ .$$

Note that $\phi+\psi = e^{\alpha-\beta+(n-1)\gamma}$; chasing through the definitions one obtains $\phi = \frac{dr}{d\hat r}\left(\frac{r}{\hat r}\right)^{n-3}$, leading eventually to the following form of (1.2.31)

$$\frac{d}{dr}\left[r^{n-1}\left(1-\frac{2m}{r^{n-2}}\right)\frac{d\hat r}{dr}\right] = (n-1)r^{n-3}\hat r\ .$$

Introducing $x=1/r$, one obtains an equation with a Fuchsian singularity at $x=0$:

$$\frac{d}{dx}\left[x^{3-n}\left(1-2mx^{n-2}\right)\frac{d\hat r}{dx}\right] = (n-1)x^{1-n}\hat r\ .$$

The characteristic exponents are -1 and $n-1$ so that, after matching a few leading coefficients, the standard theory of such equations provides solutions with the behavior

$$\hat r = r - \frac{m}{(n-2)r^{n-3}} + \begin{cases} \frac{m^2}{4}r^{-3}\ln r + O(r^{-5}\ln r), & n=4;\\ O(r^{5-2n}), & n\ge 5.\end{cases}\ .$$

Somewhat surprisingly, we find logarithms of r in an asymptotic expansion of $\hat r$ in dimension $n=4$. However, for $n\ge 5$ there is a complete expansion of $\hat r$ in terms of inverse powers of r, without any logarithmic terms in those dimensions.

As already hinted to in (1.2.28), higher dimensional counterparts of metrics (1.2.1) have been found by Tangherlini [70]. In space-time dimension $n+1$, the metrics take the form (1.2.1) with

$$V^2 = 1 - \frac{2m}{r^{n-2}}\ , \qquad (1.2.32)$$

and with $d\Omega^2$ — the unit round metric on S^{n-1}. The parameter m is the *Arnowitt-Deser-Misner mass* in space-time dimension four, and is proportional to that mass in higher dimensions. Assuming again $m>0$, a maximal analytic extension can be constructed by a simple modification of the calculations above, leading to a space-time with global structure identical to that of Figure 1.8 except for the replacement $2M\to(2M)^{1/(n-2)}$ there.

REMARK 1.2.1 For further reference we present a general construction of Walker [74]. We summarise the calculations already done: the starting point is a metric of the form

$$g = -Fdt^2 + F^{-1}dr^2 + h \ , \tag{1.2.33}$$

with $F = F(r)$, where h is a metric on an $(n-2)$-dimensional manifold (it is convenient to write F for V^2, as the sign of F did not play any role; similarly the metric h was irrelevant for the calculations we did above). We assume that F is defined for r in a neighborhood of $r = r_0$, at which F vanishes, with a simple zero there. Equivalently,

$$F(r_0) = 0 \ , \quad F'(r_0) \neq 0 \ .$$

Defining

$$u = t - f(r) \ , \quad v = t + f(r) \ , \quad f' = \frac{1}{F} \ , \tag{1.2.34}$$

$$\hat{u} = -\exp(-cu) \ , \quad \hat{v} = \exp(cv) \ , \tag{1.2.35}$$

one is led to the following form of the metric

$$g = -\frac{F}{c^2}\exp(-2cf(r))d\hat{u}\,d\hat{v} + h \ . \tag{1.2.36}$$

Since F has a simple zero, it factorizes as

$$F(r) = (r - r_0)H(r) \ ,$$

for a function H which has no zeros in a neighborhood of r_0. This follows immediately from the formula

$$F(r) - F(r_0) = \int_0^1 \frac{dF(t(r-r_0)+r_0)}{dt}\,dt = (r - r_0)\int_0^1 F'(t(r-r_0)+r_0)\,dt \ . \tag{1.2.37}$$

Now,

$$\frac{1}{F(r)} = \frac{1}{H(r_0)(r-r_0)} + \frac{1}{F(r)} - \frac{1}{H(r_0)(r-r_0)} = \frac{1}{H(r_0)(r-r_0)} + \frac{H(r_0)-H(r)}{H(r)H(r_0)(r-r_0)} \ .$$

An analysis of $H(r) - H(r_0)$ as in (1.2.37) followed by integration lead subsequently to

$$f(r) = \frac{1}{F'(r_0)}\ln|r - r_0| + \hat{f}(r) \ ,$$

for some function $\hat{f}$ which is smooth near r_0. Inserting all this into (1.2.36) with $c = F'(r_0)/2$ gives

$$g = \mp\frac{4H(r)}{(F'(r_0))^2}\exp(-\hat{f}(r)F'(r_0))d\hat{u}\,d\hat{v} + h \ , \tag{1.2.38}$$

with a positive sign if we started in the region $r > r_0$, and negative otherwise.

The function r is again implicitly defined by the equation

$$\hat{u}\hat{v} = \mp(r - r_0)\exp(\hat{f}(r)F'(r_0)) \ .$$

The right-hand-side has a derivative which equals $\mp\exp(\hat{f}(r_0)/F'(r_0)) \neq 0$ at r_0, and therefore this equation defines a smooth function $r = r(\hat{u}\hat{v})$ for r near r_0 by the implicit function theorem.

The above discussion applies to F which are of C^k differentiability class, with some losses of differentiability. Indeed, (1.2.38) provides an extension of C^{k-2} differentiability class, which leads to the restriction $k \geq 2$. However, the implicit function argument just given requires h to be differentiable, so we need in fact $k \geq 3$ for a coherent analysis. Note that for real analytic F's the extension so constructed is real analytic; this follows from the analytic version of the implicit function theorem.

Supposing we start with a region where $r > r_0$, with F positive there. Then we are in a situation reminiscent of that we encountered with the $3+1$ dimensional Schwarzschild metric, where a single region of the type I in Figure 1.4 leads to the attachment of *three* new regions to the initial manifold, through "a lower left horizon, and an upper left horizon, meeting at a corner". On the other hand, if we start with $r < r_0$ and F is negative there, we are in the situation of Figure 1.4 where a region of type II is extended through "an upper left horizon, and an upper right horizon, meeting at a corner". The reader should have no difficulties examining all remaining possibilities.

The function f of (1.2.34) for a (4+1)–dimensional Schwarzschild-Tangherlini solution can be calculated to be

$$f = r + \sqrt{2m}\ln\left(\frac{r-\sqrt{2m}}{r+\sqrt{2m}}\right) .$$

A direct calculation leads to

$$g = -\frac{8m(r+\sqrt{2m})^2}{r^2}\exp(-r/2m)\,d\hat{u}\,d\hat{v} + d\Omega^2 . \qquad (1.2.39)$$

One can similarly obtain (non-very-enlightening) explicit expressions in dimension $(5+1)$.

The isotropic coordinates in higher dimensions lead to the following form of the Schwarzschild-Tangherlini metric [60]:

$$g_m = \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}}\left(\sum_{1=1}^{n}(dx^i)^2\right) - \left(\frac{1-m/2|x|^{n-2}}{1+m/2|x|^{n-2}}\right)^2 dt^2 . \qquad (1.2.40)$$

The radial coordinate $|x|$ in (1.2.40) is related to the radial coordinate r of (1.2.32) by the formula

$$r = \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{2}{n-2}}|x| .$$

It may be considered unsatisfactory that the function r appearing in the globally regular form of the metric (1.2.16) is not given by an explicit elementary function of the coordinates. Here is a an explicit form of the extended Schwarzschild metric due to Israel [26][6]

$$g = -8m\left[dxdy + \frac{y^2}{xy+2m}dx^2\right] - (xy+2m)^2 d\Omega^2 . \qquad (1.2.41)$$

[6]The Israel coordinates have been found independently in [58], see also [31].

The coordinates (x, y) are related to the standard Schwarzschild coordinates (t, r) as follows:

$$\begin{aligned} r &= xy + 2m \ , &(1.2.42)\\ t &= xy + 2m(1 + \ln |y/x|) \ , &(1.2.43)\\ |x| &= \sqrt{|r-2m|}\exp\left(\frac{r-t}{4m}\right) \ , &(1.2.44)\\ |y| &= \sqrt{|r-2m|}\exp\left(\frac{t-r}{4m}\right) \ . &(1.2.45)\end{aligned}$$

In higher dimensions one also has an explicit, though again not very enlightening, manifestly globally regular form of the metric [35], in space-time dimension $n+1$:

$$\begin{aligned} ds^2 &= -2\,\frac{w^2(-(r)^{-n+2}2^{n+1}m^{n+1} + 4m^2((n+1)(2m-r)+3r-4m)}{m(2\,m-r)^2}dU^2 \\ &\quad +8mdUdw + r^2 d\Omega^2_{n-1} \ , \qquad (1.2.46)\end{aligned}$$

where $r \geq 0$ is the function

$$r(U,w) \equiv 2\,m + (n-2)Uw, \qquad (1.2.47)$$

while $d\Omega^2_{n-1}$ is the metric of a unit round $n-1$ sphere.

1.2.3 Some geodesics

The geodesics in the Schwarzschild metric have been studied extensively in the literature, so we will only make a few general comments about those.

First, we already encountered a family of outgoing and incoming radial null geodesics $t \mp (r + 2m\ln(r-2m)) = \text{const.}$

Next, each Killing vector X produces a constant of motion $g(X,\dot\gamma)$ along an affinely parameterised geodesic. So we have a conserved energy

$$\mathscr{E} := g(\partial_t, \dot\gamma) = -(1-\frac{2m}{r})\dot t \ ,$$

and a conserved angular momentum ω

$$\omega := g(\partial_\varphi, \dot\gamma) = r^2\dot\varphi \ .$$

Yet another constant of motion arises from the length of $\dot\gamma$,

$$g(\dot\gamma,\dot\gamma) = -(1-\frac{2m}{r})\dot t^2 + \frac{\dot r^2}{1-\frac{2m}{r}} + r^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) = \varepsilon \in \{-1,0,1\} \ . \quad (1.2.48)$$

To simplify things somewhat, let us show that all motions are *planar*. One way of doing this is to write the equations explicitly. The Lagrangean for geodesics reads:

$$\mathscr{L} = \frac{1}{2}\left(V^2\left(\frac{dt}{ds}\right)^2 - V^{-2}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right) \ .$$

Those Euler-Lagrange equations which are not already covered by the conservation laws read:

$$\frac{d}{ds}\left(V^{-2}\frac{dr}{ds}\right) = V\partial_r V\left(\frac{dt}{ds}\right)^2 + 2r\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right], \quad (1.2.49)$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = r^2\sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 . \quad (1.2.50)$$

Consider any geodesic, and think of the coordinates (r,θ,φ) as spherical coordinates on $\mathbb{R}^3$. Then the initial position vector (which is, for obvious reasons, assumed *not* to be the origin) and the initial velocity vector, which is assumed *not* to be radial (otherwise the geodesic will be radial, and the claim follows) define a unique plane in $\mathbb{R}^3$. We can then choose the spherical coordinates so that this plane is the plane $\theta = \pi/2$. We then have $\theta(0) = \pi/2$ and $\dot\theta(0) = 0$, and then $\theta(s) \equiv \pi/2$ is a solution of (1.2.50) satisfying the initial values. By uniqueness this is *the* solution.

So, without loss of generality we can assume $\sin\theta = 1$ throughout the motion, from (1.2.48) we then obtain the following ODE for $r(s)$;

$$\dot r^2 = \mathscr{E}^2 + (1-\frac{2m}{r})(\varepsilon - \frac{\omega^2}{r^2}) . \quad (1.2.51)$$

The radial part of the geodesic equation can be obtained by calculating directly the Christoffel symbols of the metric. A more efficient way is to use the variational principle for geodesics, with the Lagrangean $\mathscr{L} = g(\dot\gamma,\dot\gamma)$ — this can be read off from the middle term in (1.2.48). But the reader should easily convince herself that, at this stage, the desired equation can be obtained by differentiating (1.2.51) with respect to s, obtaining

$$2\frac{d^2r}{ds^2} = \frac{d}{dr}\Big(\mathscr{E}^2 + (1-\frac{2m}{r})(\varepsilon - \frac{\omega^2}{r^2})\Big) . \quad (1.2.52)$$

We wish to point out the existence of a striking class of *null* geodesics for which $r(s) = \text{const}$. It follows from (1.2.52), and from uniqueness of solutions of the Cauchy problem for ODE's, that such a curve will be a null geodesic provided that the right-hand-sides of (1.2.51) and of (1.2.52) (with $\varepsilon = 0$) vanish:

$$\mathscr{E}^2 - (1-\frac{2m}{r})\frac{\omega^2}{r^2} = 0 = \frac{2\omega^2}{r^3}(-r+3m) . \quad (1.2.53)$$

Simple algebra shows now that the curves

$$s \mapsto \gamma_\pm(s) = (t = s,\ r = 3m,\ \theta = \pi/2,\ \varphi = \pm 3^{3/2}m^{-1}s) ,$$

are thus null geodesics spiraling on the timelike cylinder $\{r = 3m\}$.

1.2.4 The Flamm paraboloid

We write again the Schwarzschild metric in dimension $n+1$,

$$g_m = -\left(1-\frac{2m}{r^{n-2}}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r^{n-2}}} + r^2 d\Omega^2 , \quad (1.2.54)$$

where, as usual, $d\Omega^2$ is the round unit metric on S^{n-1}. Because of spherical symmetry, the geometry of the $t = const$ slices can be realised by an embedding into $(n+1)$–dimensional Euclidean space. If we set

$$\mathring{g} = dz^2 + (dx^1)^2 + \ldots + (dx^n)^2 = dz^2 + dr^2 + r^2 d\Omega^2 ,$$

the metric h induced by $\mathring{g}$ on the the surface $z = z(r)$ reads

$$h = \Big(\Big(\frac{dz}{dr}\Big)^2 + 1 \Big) dr^2 + r^2 d\Omega^2 .$$

This will coincide with the space part of (1.2.54) if we require that

$$\frac{dz}{dr} = \pm \sqrt{\frac{2m}{r^{n-2} - 2m}} .$$

The equation can be explicitly integrated in dimensions $n = 3$ and 4 in terms of elementary functions, leading to

$$z = z_0 \pm \sqrt{2m} \times \begin{cases} 2\sqrt{r - 2m}, & r > 2m,\ n = 3, \\ \ln(r + \sqrt{r^2 - 2m}), & r > \sqrt{2m},\ n = 4. \end{cases}$$

The positive sign corresponds to the usual black hole exterior, while the negative sign corresponds to the second asymptotically flat region, on the "other side" of the Einstein-Rosen bridge. Solving for $r(z)$, a convenient choice of z_0 leads to

$$r = \begin{cases} 2m + z^2/8m, & n = 3, \\ \sqrt{2m}\cosh(z/\sqrt{2m}), & n = 4. \end{cases}$$

In dimension $n = 3$ one obtains a paraboloid, as first noted by Flamm. The embeddings are visualized in Figures 1.5 and 1.6.

The qualitative behavior in dimensions $n \geq 5$ is somewhat different, as then $z(r)$ asymptotes to a finite value as r tends to infinity. The embeddings in $n = 5$ are visualized in Figure 1.7; in that dimension $z(r)$ can be expressed in terms of elliptic functions, but the final formula is not very illuminating.

1.2.5 Conformal Carter-Penrose diagrams

Consider a metric with the following product structure:

$$g = \underbrace{g_{rr}(t,r)dr^2 + 2g_{rt}(t,r)dtdr + g_{tt}(t,r)dr^2}_{=:{}^2g} + \underbrace{h_{AB}(t,r,x^A)dx^A dx^B}_{=:h} , \quad (1.2.55)$$

where h is Riemannian metric in dimension $n-1$. Then any causal vector for g is also a causal vector for ${}^2g$, and drawing light-cones for ${}^2g$ gives a good idea of the causal structure of $(\mathscr{M}, g)$. We have already done that in Figure 1.4 to depict the black hole character of the Kruskal-Szekeres space-time.

Now, it is not too difficult to prove that any two-dimensional metric can be brought locally to the form

$${}^2g = 2g_{uv}(u,v)dudv = 2g_{uv}(-dt^2 + dr^2) \quad (1.2.56)$$

in which the light-cones have slopes one, just as in Minkowski space-time. When using such coordinates, it is sufficient to draw their domain of definition to visualise the global causal structure of the space-time.

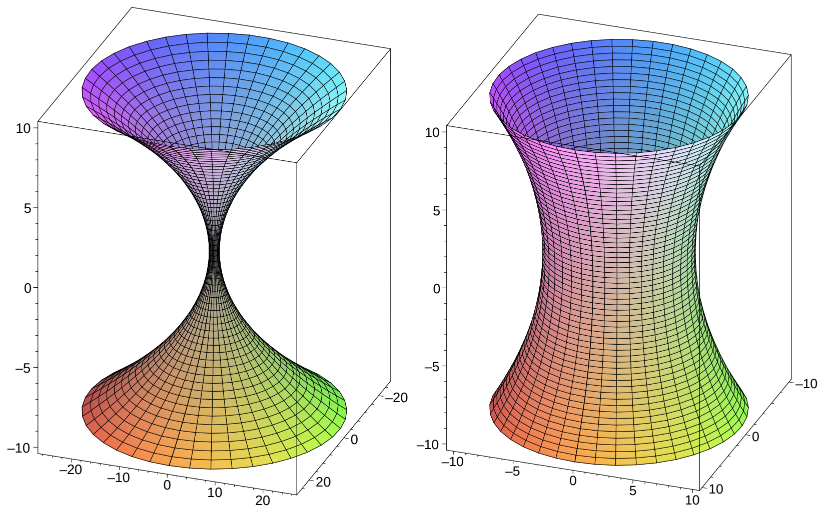

Figure 1.5: Isometric embedding of the space-geometry of an $n = 3$ dimensional Schwarzschild black hole into four-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = 2m$, with $2m = 1$ (left) and $2m = 6$ (right).

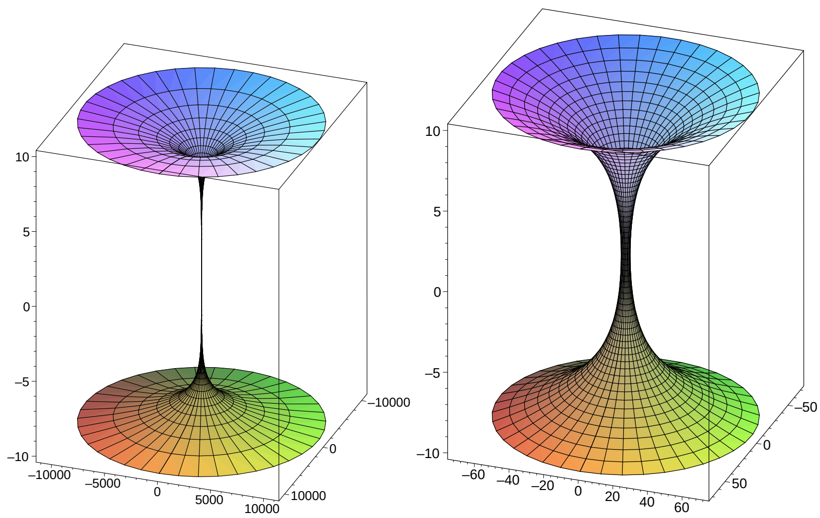

Figure 1.6: Isometric embedding of the space-geometry of an $n = 4$ dimensional Schwarzschild black hole into five-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = (2m)^{1/2}$, with $2m = 1$ (left) and $2m = 6$ (right). The extents of the vertical axis are the same as those in Figure 1.5.

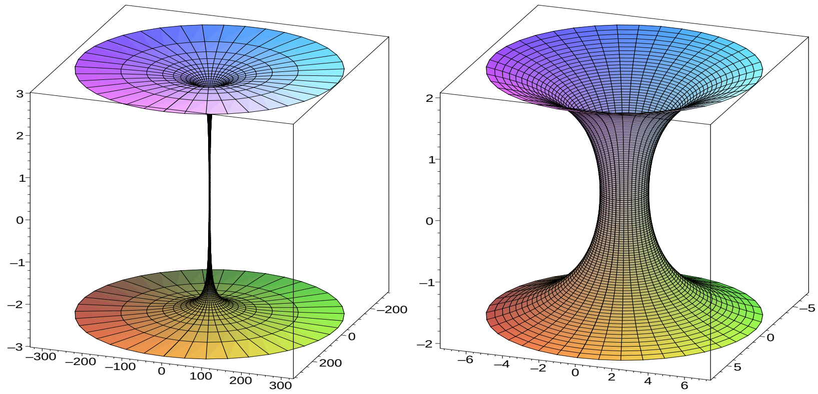

Figure 1.7: Isometric embedding of the *space-geometry* of a $(5+1)$–dimensional Schwarzschild black hole into six-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = (2m)^{1/3}$, with $2m = 2$. The variable along the vertical axis asymptotes to $\approx \pm 3.06$ as r tends to infinity. The right picture is a zoom to the centre of the throat.

EXERCICE 1.2.2 Prove (1.2.56). *[Hint: use coordinates associated with right-going and left-going null geodesics.]*

The above are the first two-ingredients behind the idea of conformal Carter-Penrose diagrams. The last thing to do is to bring any infinite domain of definition of the (u, v) coordinates to a finite one. An essentially identical coordinate transformation works here: Indeed, let $\bar{u}$ and $\bar{v}$ be defined by the equations

$$\tan \bar{u} = \frac{\hat{u}}{\sqrt{2m}} \ , \quad \tan \bar{v} = \frac{\hat{v}}{\sqrt{2m}} \ ,$$

where $\hat{v}$ and $\hat{u}$ have been defined in (1.2.14)-(1.2.15). Using

$$d\hat{u} = \frac{\sqrt{2m}}{\cos^2 \bar{u}} d\bar{u} \ , \quad d\hat{v} = \frac{\sqrt{2m}}{\cos^2 \bar{v}} d\bar{v} \ ,$$

the Schwarzschild metric (1.2.16) takes the form

$$\begin{aligned} g &= -\frac{16m^2 \exp(-\frac{r}{2m})}{r} d\hat{u}\, d\hat{v} + r^2 d\Omega^2 \\ &= -\frac{32m^3 \exp(-\frac{r}{2m})}{r \cos^2 \bar{u} \cos^2 \bar{v}} d\bar{u}\, d\bar{v} + r^2 d\Omega^2 \ . \end{aligned} \tag{1.2.57}$$

Introducing new time- and space-coordinates $\bar{t} = (\bar{u} + \bar{v})/2$, $\bar{x} = (\bar{u} - \bar{v})/2$, so that

$$\bar{u} = \bar{t} - \bar{x} \ , \quad \bar{v} = \bar{t} + \bar{x} \ ,$$

one obtains a more familiar-looking form

$$g = -\frac{32m^3 \exp(-\frac{r}{2m})}{r \cos^2 \bar{u} \cos^2 \bar{v}} (-d\bar{t}^2 + d\bar{x}^2) + r^2 d\Omega^2 \ .$$

This is regular except at $\cos\bar u = 0$, and $\cos\bar u = 0$, and $r = 0$. The first set corresponds to the straight lines $\bar u = \bar t - \bar x \in \{\pm\pi/2\}$, while the second is the union of the lines $\bar v = \bar t + \bar x \in \{\pm\pi/2\}$.

The analysis of $\{r = 0\}$ requires some work: recall that $r = 0$ corresponds to $\hat u \hat v = 2m$, which is equivalent to

$$\tan(\bar u)\tan(\bar v) = 1\ .$$

Using the formula

$$\tan(\bar u + \bar v) = \frac{\tan\bar u + \tan\bar v}{1 - \tan\bar u \tan\bar v}$$

we obtain "$\tan(\bar u + \bar v) = \pm\infty$" or, more precisely,

$$\bar u + \bar v = 2\bar t = \pm\pi/2\ .$$

So the Kruskal-Szekeres metric is conformal to a smooth Lorentzian metric on $C \times S^2$, where C is the set of Figure 1.8.

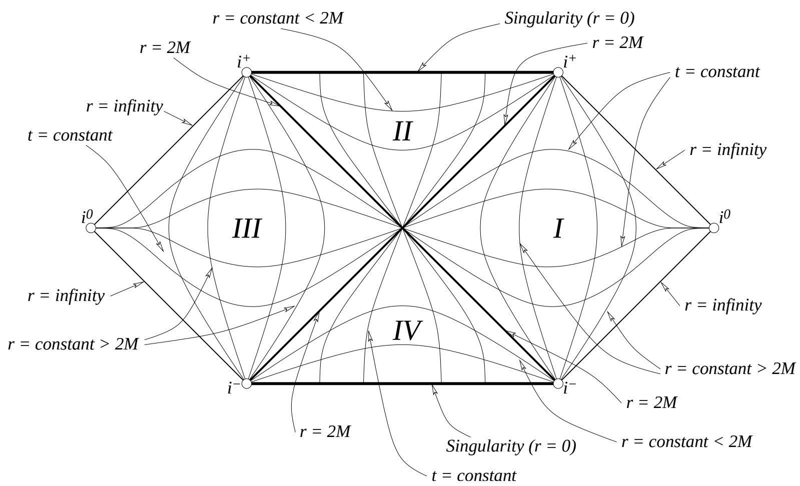

Figure 1.8: The Carter-Penrose diagram[5] for the Kruskal-Szekeres space-time with mass M. There are actually two asymptotically flat regions, with corresponding event horizons defined with respect to the second region. Each point in this diagram represents a two-dimensional sphere, and coordinates are chosen so that light-cones have slopes plus minus one. Regions are numbered as in Figure 1.4.

1.3 Some general notions

Before continuing some general notions are in order. A *Killing field*, by definition, is a vector field the local flow of which preserves the metric. One of the features of the metric (1.2.1) is its *stationarity*, with Killing vector field $X = \partial_t$: A space–time is called *stationary* if there exists a Killing vector field X which approaches ∂_t in the asymptotically flat region (where r goes to ∞, see Section 2.3 below for precise definitions) *and* generates a one parameter groups of isometries. A space–time is called *static* if it is stationary and if the stationary Killing vector X is hypersurface-orthogonal, i.e.

$$X^\flat \wedge dX^\flat = 0 ,$$

where

$$X^\flat = X_\mu dx^\mu = g_{\mu\nu} X^\nu dx^\mu .$$

EXERCICE 1.3.1 Show that the Schwarzschild and the Reissner-Nordström metrics are static, but that the Kerr metrics with $a \neq 0$ are not.

A space–time is called *axisymmetric* if there exists a Killing vector field Y, which generates a one parameter group of isometries, and which behaves like a *rotation*: this property is captured by requiring that all orbits 2π periodic, and that the set $\{Y = 0\}$, called the *axis of rotation*, is non-empty. Killing vector fields which are a non-trivial linear combination of a time translation and of a rotation in the asymptotically flat region are called *stationary-rotating*, or *helical*. Note that those definitions require completeness of orbits of all Killing vector fields (this means that the equation $\dot{x} = X$ has a global solution for all initial values), see Refs. [11] and [21] for some results concerning this question.

In the extended Schwarzschild space-time the set $\{r = 2m\}$ is a null hypersurface $\mathscr{E}$, the Schwarzschild event horizon. The stationary Killing vector $X = \partial_t$ extends to a Killing vector $\hat{X}$ in the extended spacetime which becomes tangent to and null on $\mathscr{E}$, except at the "bifurcation sphere" right in the middle of Figure 1.8, where $\hat{X}$ vanishes.

A null hypersurface which coincides with a connected component of the set

$$\mathscr{N}_X := \{g(X,X) = 0 \ , \ X \neq 0\} ,$$

where X is a Killing vector, is called a *Killing horizon* associated to X. Figure 1.4 makes it clear that the event horizon $\{r = 2m\}$ of the Kruskal-Szekeres space-time is the union of four Killing horizons and of the bifurcation surface, with respect to the Killing vector field which equals ∂_t in the asymptotically flat region.

Another similar example is provided by the "boost Killing vector field"

$$K = z\partial_t + t\partial_z \tag{1.3.1}$$

in Minkowski space-time: The Killing horizon $\mathscr{N}(K)$ of K has four connected components

$$\mathscr{N}(K)_{\epsilon\delta} := \{t = \epsilon z \ , \delta t > 0\} , \quad \epsilon, \delta \in \{\pm 1\} .$$

The closure $\overline{\mathscr{N}(K)}$ of $\mathscr{N}(K)$ is the set $\{|t| = |z|\}$, which is *not* a manifold, because of the crossing of the null hyperplanes $\{t = \pm z\}$ at $t = z = 0$. Horizons of this type are referred to as *bifurcate Killing horizons*.

One more noteworthy example, in Minkowski space-time, is provided by the Killing vector

$$X = y\partial_t + t\partial_y + x\partial_y - y\partial_x = y\partial_t + (t+x)\partial_y - y\partial_x . \tag{1.3.2}$$

Thus, X is the sum of a boost $y\partial_t + t\partial_y$ and a rotation $x\partial_y - y\partial_x$. Note that X vanishes if and only if

$$y = t + x = 0 ,$$

which is a two-dimensional isotropic submanifold of Minkowski space-time $\mathbb{R}^{1,3}$. Further,

$$g(X, X) = (t + x)^2 = 0$$

which is an isotropic hyperplane in $\mathbb{R}^{1,3}$.

1.3.1 Surface gravity

The *surface gravity* κ of a Killing horizon is defined by the formula

$$(X^\alpha X_\alpha)_{,\mu}\Big|_{\mathscr{N}(X)} = -2\kappa X_\mu \ . \tag{1.3.3}$$

A word of justification is in order here: since $g(X, X) = 0$ on $\mathscr{N}(X)$ the differential of $g(X, X)$ is conormal to $\mathscr{N}(X)$. Recalling that on a null hypersurface the conormal is proportional to $g(\ell, \cdot)$, where ℓ is any null vector tangent to $\mathscr{N}$ (those are defined uniquely up to a proportionality factor), we obtain that $d(g(X, X))$ is proportional to $X^\flat = X_\mu dx^\mu$; whence (1.3.3).

As an example, consider the Killing vector K of (1.3.1). We have

$$d(g(K, K)) = d(-z^2 + t^2) = 2(-zdz + tdt) ,$$

which is twice $K^\flat$ on $\mathscr{N}(K)_{\epsilon\delta}$. On another hand, for the Killing vector X of (1.3.2) we have

$$d(g(X, X)) = 2(t + x)(dt + dx) ,$$

which vanishes on each of the Killing horizons $\{t = -x \ , y \neq 0\}$.

The surface gravity of black holes plays an important role in *black hole thermodynamics*, cf., e.g., [3] and references therein.

A Killing horizon $\mathscr{N}(X)$ is said to be *degenerate*, or *extreme*, if κ vanishes throughout $\mathscr{N}(X)$; it is called *non-degenerate* if κ has no zeros on $\mathscr{N}(X)$. Thus, the Killing horizons $\mathscr{N}(K)_{\epsilon\delta}$ are non-degenerate, while both Killing horizons of X given by (1.3.2) are degenerate.

EXAMPLE 1.3.2 Consider the Schwarzschild metric in the representation (1.2.6),

$$g = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \ . \tag{1.3.4}$$

We have

$$d\Big(g(X, X)\Big) = d\Big(g(\partial_v, \partial_v)\Big) = -\frac{2m}{r^2}dr \ .$$

Now, $X^\flat = g(\partial_v, \cdot) = -(1 - \frac{2m}{r})dv + dr$, which equals dr for $r = 2m$. Comparing with (1.3.3) gives

$$\kappa \equiv \kappa_m := \frac{1}{2m} \ .$$

We see that the Schwarzschild black holes are all non-degenerate, with surface gravity $(2m)^{-1}$. So there are no degenerate black holes within the Schwarzschild family. It will be seen in Section 2.5.2 that there are no regular, degenerate, static vacuum black holes at all.

In Kerr space-times (see Section 1.5 below) we have $\kappa = 0$ if and only if $m = a$. On the other hand, all horizons in the multi-black hole Majumdar-Papapetrou solutions of Section 1.6 are degenerate.

The surface gravity κ is constant on bifurcate [29, p. 59] Killing horizons.

Yet another class of space-times with constant κ ([24], Theorem 7.1) is provided by space-times satisfying the *dominant energy condition*: this means that $T_{\mu\nu}X^\mu Y^\nu \geq 0$ for all timelike future directed vector fields X and Y.

1.4 The Reissner-Nordström metrics

The Reissner-Nordström metrics are the unique spherically symmetric solutions of the Einstein-Maxwell equations (with vanishing cosmological constant). They turn out to be static, asymptotically flat, and describe black hole space-times with interesting global properties for a certain range of parameters. The metric takes the form

$$ {}^4g = -\Big(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2 \ , \tag{1.4.1} $$

where m is, as usual, the ADM mass of g and Q is the total electric charge. The electromagnetic potential takes the form

$$ A = \frac{Q}{r} dr \ . \tag{1.4.2} $$

The equation $g(\partial_t, \partial_t) = 0$ has solutions $r = r_\pm$ provided that $|Q| \leq m$:

$$ r_\pm = m \pm \sqrt{m^2 - Q^2} \ . $$

Calculating as in Example 1.3.2, one finds that the surface gravities of the horizons $r = r_\pm$ of the Reissner-Nordström metric equal

$$ \begin{aligned} \kappa_\pm &= -\frac{1}{2}\partial_r g_{tt}|_{r=r_\pm} = \frac{1}{2}\partial_r\Big(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\Big)\Big|_{r=r_\pm} = \frac{mr_\pm - Q^2}{r_\pm^3} \\ &= \pm\frac{\sqrt{m^2 - Q^2}}{r_\pm^2} \ . \end{aligned} $$

For $r = r_+$ this is strictly positive unless $|Q| = m$; so we see that Reissner-Nordström black holes are non-degenerate for $|Q| < m$, and degenerate when $|Q| = m$.

In dimensions $n + 1 \geq 5$ one has [52] the following counterpart of (1.4.1)-(1.4.2):

$$ {}^{n+1}g = -\Big(1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}\Big)dt^2 + \frac{dr^2}{1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}} + r^2 d\Omega^2 \ , \tag{1.4.3} $$

$$ A = \frac{Q}{r^{n-2}} dr \ , \tag{1.4.4} $$

where m is related to the ADM mass, and Q to the total charge.

1.5 The Kerr metric

There is a rotating generalisation of the Schwarzschild metric, namely the two parameter family of *exterior Kerr metrics*, which in Boyer-Lindquist coordinates take the form

$$g = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{4amr\sin^2\theta}{\Sigma}dtd\varphi + \\ + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \ . \qquad (1.5.1)$$

Here

$$\Sigma = r^2 + a^2\cos^2\theta \ , \qquad \Delta = r^2 + a^2 - 2mr = (r - r_+)(r - r_-) \ ,$$

and $r_+ < r < \infty$, where

$$r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}} \ .$$

The metric satisfies the vacuum Einstein equations for any real values of the parameters a and m, but we will only discuss the range $0 \le a < m$. When $a = 0$, the Kerr metric reduces to the Schwarzschild metric. The Kerr metric is again a vacuum solution, and it is stationary with $X = \partial_t$ the asymptotic time translation, as well as axisymmetric with $Y = \partial_\varphi$ the generator of rotations. Similarly to the Schwarzschild case, it turns out that the metric can be smoothly extended across $r = r_+$, with $\{r = r_+\}$ being a smooth null hypersurface $\mathscr{E}$ in the extension. The simplest extension is obtained when t is replaced by a new coordinate

$$v = t + \int_{r_+}^{r} \frac{r^2+a^2}{\Delta}dr \ , \qquad (1.5.2)$$

with a further replacement of φ by

$$\phi = \varphi + \int_{r_+}^{r} \frac{a}{\Delta}dr \ . \qquad (1.5.3)$$

It is convenient to use the symbol $\hat{g}$ for the metric g in the new coordinate system, obtaining

$$\begin{aligned} \hat{g} \ &= \ -\Big(1 - \frac{2mr}{\Sigma}\Big)dv^2 + 2drdv + \Sigma d\theta^2 - 2a\sin^2\theta d\phi dr \\ &\quad + \frac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta}{\Sigma}\sin^2\theta d\phi^2 - \frac{4amr\sin^2\theta}{\Sigma}d\phi dv \ . \qquad (1.5.4)\end{aligned}$$

In order to see that (1.5.4) provides a smooth Lorentzian metric for $v \in \mathbb{R}$ and $r \in (0, \infty)$, note first that the coordinate transformation (1.5.2)-(1.5.3) has been tailored to remove the $1/\Delta$ singularity in (1.5.1), so that all coefficients are now analytic functions on $\mathbb{R} \times (0, \infty) \times S^2$. A direct calculation of the determinant of $\hat{g}$ is somewhat painful, a simpler way is to proceed as follows: first, the calculation of the determinant of the metric (1.5.1) reduces to that of a two-by-two determinant in the (t, ψ) variables, leading to $\det g = -\sin^2\theta\Sigma^2$. Next, it is very easy to check that the determinant of the Jacobi matrix $\partial(v, r, \theta, \phi)/\partial(t, r, \theta, \varphi)$ is one. It follows that $\det \hat{g} = -\sin^2\theta\Sigma^2$ for $r > r_+$. Analyticity implies that this equation holds globally, which (since Σ has no zeros) establishes the Lorentzian signature of $\hat{g}$ for all positive r.

Let us show that the region $r < r_+$ is a black hole region, in the sense of (1.2.8). We start by noting that ∇r is a causal vector for $r_- \leq r \leq r_+$, where $r_- = m - \sqrt{m^2 + a^2}$. A direct calculation using (1.5.4) is again somewhat lengthy, instead we use (1.5.1) in the region $r > r_+$ to obtain there

$$\hat{g}(\nabla r, \nabla r) = g(\nabla r, \nabla r) = g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\Sigma} = \frac{(r - r_+)(r - r_-)}{r^2 + a^2\cos^2\theta} \, . \tag{1.5.5}$$

But the left-hand-side of this equation is an analytic function throughout the extended manifold $\mathbb{R}\times(0,\infty)\times S^2$, and uniqueness of analytic extensions implies that $\hat{g}(\nabla r, \nabla r)$ equals the expression at the extreme right of (1.5.5). (The intermediate equalities are of course valid only for $r > r_+$.) Thus ∇r is spacelike if $r < r_-$ or $r > r_+$, null on the *"Killing horizons"* $\{r = r_\pm\}$, and timelike in the region $\{r_- < r < r_+\}$. We choose a time orientation so that ∇r is future pointing there.

Consider, now, a future directed causal curve $\gamma(s)$. Along γ we have

$$\frac{dr}{ds} = \dot{\gamma}^i \nabla_i r = g_{ij}\dot{\gamma}^i \nabla^j r = g(\dot{\gamma}, \nabla r) < 0 \tag{1.5.6}$$

in the region $\{r_- < r < r_+\}$, because the scalar product of two future directed causal vectors is always negative. This implies that r is strictly decreasing along future directed causal curves in the region $\{r_- < r < r_+\}$, so that such curves can only leave this region through the set $\{r = r_-\}$. In other words, no causal communication is possible from the region $\{r < r_+\}$ to the "exterior world" $\{r > r_+\}$.

The Schwarzschild metric has the property that the set $g(X, X) = 0$, where X is the "static Killing vector" ∂_t, coincides with the event horizon $r = 2m$. This is not the case any more for the Kerr metric, where we have

$$g(\partial_t, \partial_t) = \hat{g}(\partial_v, \partial_v) = \hat{g}_{vv} = -\Big(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\Big) \, ,$$

and the equation $\hat{g}(\partial_v, \partial_v) = 0$ defines a set called the *ergosphere*:

$$\mathring{r}_\pm = m \pm \sqrt{m^2 - a^2\cos^2\theta} \, ,$$

see Figures 1.9 and 1.10. The ergosphere touches the horizons at the axes of symmetry $\cos\theta = \pm 1$. Note that $\partial\mathring{r}_\pm/\partial\theta \neq 0$ at those axes, so the ergosphere has a cusp there. The region bounded by the outermost horizon $r = r_+$ and the outermost ergosphere $r = \mathring{r}_+$ is called the *ergoregion*, with X spacelike in its interior. We refer the reader to Refs. [7] and [56] for an exhaustive analysis of the geometry of the Kerr space-time.

One of the most useful methods for analysing solutions of wave equations is the *energy method*. As an illustration, consider the wave equation

$$\Box u = 0 \, . \tag{1.5.7}$$

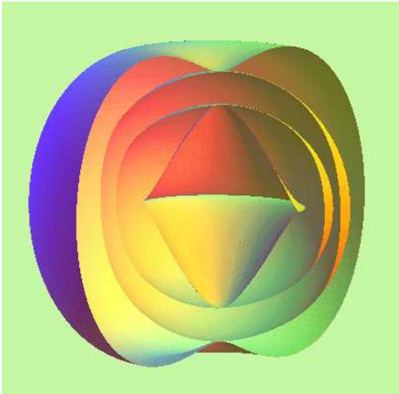

Figure 1.9: A coordinate representation [62] of the outer ergosphere $r = \mathring{r}_+$, the event horizon $r = r_+$, the Cauchy horizon $r = r_-$, and the inner ergosphere $r = \mathring{r}_-$ with the singular ring in Kerr space-time. Computer graphics by Kayll Lake [34].

Let $\mathscr{S}_t$ is a foliation of $\mathscr{M}$ by spacelike hypersurfaces, the energy E_t of u on $\mathscr{S}_t$ associated to a vector field X is defined as

$$E(t) = \int_{\mathscr{S}_t} T^\mu{}_\nu X^\mu \eta_\nu \ ,$$

where $T_{\mu\nu}$ is the usual energy-momentum tensor of a scalar field,

$$T_{\mu\nu} = \nabla_\mu u \nabla_\nu u - \frac{1}{2}\nabla^\alpha u \nabla_\alpha u \, g_{\mu\nu} \ .$$

The energy functional E has two important properties: 1): $E \geq 0$ if X is causal, and 2): $E(t)$ is conserved if X is a Killing vector field and, say, u has compact support on each of the $\mathscr{S}_t$.

Now, the existence of ergoregions where the Killing vector X becomes space-like leads to an $E(t)$ which is not necessarily positive any more, and the energy stops being a useful tool in controlling the behavior of the field. This is one of the obstactles to our understanding of both linear and non-linear, solutions of wave equations on a Kerr background[7], not to mention the wide open question of non-linear stability of the Kerr black holes within the class of globally hyperbolic solutions of the vacuum Einstein equations.

The hypersurfaces

$$\mathscr{H}_\pm := \{r = r_\pm\}$$

provide examples of *null acausal boundaries*. Because $g(\nabla r, \nabla r)$ vanishes at $\mathscr{H}_\pm$, the usual calculation shows that the integral curves of ∇r with $r = r_\pm$ are null geodesics. Such geodesics, tangent to a null hypersurface, are called *generators* of this hypersurface. A direct calculation of ∇r from (1.5.4) requires work which can be avoided as follows: in the coordinate system (t, r, θ, φ) of (1.5.1) one obtains immediately

$$\nabla r = g^{\mu\nu}\partial_\mu r \partial_\nu = \frac{\Delta}{\Sigma}\partial_r \ .$$

[7]See [2] and refs. therein for further information on that subject.

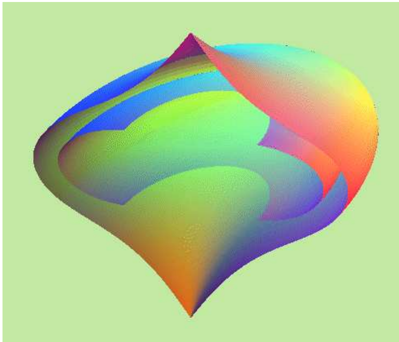

Figure 1.10: Isometric embedding in Euclidean three space of the ergosphere (the outer hull), and part of the event horizon, for a rapidly rotating Kerr solution. The hole in the event horizon arises because there is no global isometric embedding for the event horizon when $a/m > \sqrt{3}/2$ [62]. Somewhat surprisingly, the embedding fails to represent accurately the fact that the cusps at the rotation axis are pointing inwards, and not outwards. Computer graphics by Kayll Lake [34].

Now, under (1.5.2)-(1.5.3) the vector ∂_r transforms as

$$\partial_r \to \partial_r + \frac{a}{\Delta}\partial_\phi + \frac{r^2+a^2}{\Delta}\partial_v \ .$$

This shows that in the coordinates (v, r, θ, ϕ) we have

$$\nabla r = \frac{\Delta}{\Sigma}\partial_r + a\partial_\phi + (r^2+a^2)\partial_v \ .$$

Since Δ vanishes at $r = r_\pm$, and $r^2 + a^2$ equals $2mr_\pm$ there, we conclude that the "stationary-rotating" Killing field $X + \omega Y$, where

$$X = \partial_t = \partial_v \ , \quad Y = \partial_\phi = \partial_\varphi \ , \quad \omega = \frac{a}{2mr_+} \ , \tag{1.5.8}$$

is proportional to ∇r on $\{r > r_+\}$:

$$X + \omega Y = 2mr_+ \nabla r \quad \text{on} \quad \mathscr{H}_+ \ .$$

It follows that $\partial_t + \omega\partial_\varphi$ is null and *tangent* to the generators of the horizon $\mathscr{H}_+$. In other words, the generators are rotating with respect to the frame defined by the stationary Killing vector field X. This property is at the origin of the definition of ω as the *angular velocity* of the event horizon.

Higher dimensional generalisations of the Kerr metric have been constructed by Myers and Perry [52].

1.6 Majumdar-Papapetrou multi black holes

In the examples discussed so far the black hole event horizon is a connected hypersurface in space-time. In fact [4, 13], there are no *regular, static, vacuum* solutions with several black holes, consistently with the intuition that gravity is an attractive force. However, static multi black holes become possible in presence of electric fields. Well-behaved examples are exhausted [16] by the *Majumdar-Papapetrou* black holes, in which the metric ${}^4g$ and the electromagnetic potential A take the form [43, 59]

$$ {}^4g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \tag{1.6.1} $$

$$ A = u^{-1}dt\,, \tag{1.6.2} $$

with some nowhere vanishing function u. Einstein–Maxwell equations read then

$$ \frac{\partial u}{\partial t} = 0\,, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\,. \tag{1.6.3} $$

Regular, or *standard MP black holes* are obtained if the coordinates x^μ of (1.6.1)–(1.6.2) cover the range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$, and if the function u has the form

$$ u = 1 + \sum_{i=1}^{I} \frac{\mu_i}{|\vec{x} - \vec{a}_i|}\,, \tag{1.6.4} $$

for some positive constants μ_i.

The property that these are the only regular black holes within the MP class has been proved in [14], see also [15, 23]; the fact that all multi-component regular static black holes are in the MP class has been established in [16], building upon the work in [44, 65, 66].

The case $I = \infty$ has been considered in [12, Appendix B], where it was pointed out that the scalar $F_{\mu\nu}F^{\mu\nu}$ is unbounded whenever the $\vec{a}_i$'s have accumulation points. It follows from [14] that the case where $I = \infty$ and the $\vec{a}_i$'s do not have accumulation points cannot lead to regular asymptotically flat space–times.

Calculating the flux of the electric field on spheres $|\vec{x} - \vec{a}_i| = \epsilon \to 0$ one finds that μ_i is the electric charge carried by the puncture $\vec{x} = \vec{a}_i$.

Higher-dimensional generalisations of the MP solutions have been pointed out by Myers [51]. The metric and the electromagnetic potential take the form

$$ {}^{n+1}g = -u^{-2}dt^2 + u^{\frac{2}{n-2}}\Big((dx^1)^2 + \ldots + (dx^n)^2\Big)\,, \tag{1.6.5} $$

$$ A = u^{-1}dt\,, \tag{1.6.6} $$

with u being time independent, and harmonic with respect to the flat metric $(dx^1)^2 + \ldots + d(x^n)^2$. Then, a natural candidate potential u for solutions with black holes takes the form

$$ u = 1 + \sum_{i=1}^{N} \frac{\mu_i}{|\vec{x} - \vec{a}_i|^{n-2}}\,, \tag{1.6.7} $$

for some $\vec{a}_i \in \mathbb{R}^n$.

Let us point out some features of the geometries (1.6.5). First, for large $|\vec{x}|$ we have

$$u = 1 + \frac{\sum_{i=1}^N \mu_i}{|\vec{x}|^{n-2}} + O(|\vec{x}|^{-(n-1)}) ,$$

so that the metric is asymptotically flat, with total ADM mass equal to $\sum_{i=1}^N \mu_i$.

Next, choose any i and let $r = |\vec{x} - \vec{a}_i|$ be a radial coordinate centred at $\vec{a}_i$. Then the space-part g of the metric (1.6.5) takes the form

$$\begin{aligned} g &= u^{\frac{2}{n-2}}\Big((dx^1)^2 + \ldots + (dx^n)^2\Big) = r^2 u^{\frac{2}{n-2}}\Big(\frac{dr^2}{r^2} + h\Big) \\ &= (r^{\frac{1}{n-2}}u)^{\frac{2}{n-2}}(d(\underbrace{\ln r}_{=:x})^2 + h) \qquad (1.6.8) \\ &= (r^{\frac{1}{n-2}}u)^{\frac{2}{n-2}}(dx^2 + h) , \qquad (1.6.9) \end{aligned}$$

where h is the unit round metric on S^{n-1}. Now, the metric $dx^2 + h$ is the canonical, complete, product metric on the cylinder $\mathbb{R} \times S^{n-1}$. Further

$$r^{\frac{1}{n-2}}u \to_{\vec{x}\to\vec{a}_i} \mu_i > 0 .$$

Therefore the space-part of the Majumdar-Papapetrou metric approaches a multiple of the canonical metric on $\mathbb{R} \times S^{n-1}$ as $\vec{x}$ approaches $\vec{a}_i$. Hence, the space geometry is described by a complete metric which has one asymptotically flat region $|\vec{x}| \to \infty$ and N asymptotically cylindrical regions $\vec{x} \to \vec{a}_i$.

It has been shown by Hartle and Hawking [23] that, in dimension $n = 3$, every standard MP space–time can be analytically extended to an electro–vacuum space–time with I black hole regions, the calculation (keeping an eye on $n \geq 3$) proceeds as follows: Let, as before, $r = |\vec{x} - \vec{a}_i|$; for r small we replace t by a new coordinate v defined as

$$v = t + f(r) \qquad \Longrightarrow \qquad dt = dv - f'(r)dr ,$$

with a function f to be determined shortly. We obtain

$$\begin{aligned} {}^{n+1}g &= -u^{-2}(dv - f'dr)^2 + u^{\frac{2}{n-2}}(dr^2 + r^2 h) \\ &= -u^{-2}dv^2 + 2u^{-2}f'dv\,dr + \Big(u^{\frac{2}{n-2}} - u^{-2}(f')^2\Big)dr^2 + u^{\frac{2}{n-2}}r^2 h . \end{aligned} \qquad (1.6.10)$$

We have already seen that the last term $u^{\frac{2}{n-2}}r^2h$ is well behaved, let us show that in some cases we can choose f to get rid of the singularity in g_{rr}. For this we Taylor expand u near $\vec{a}_i$ as follows:

$$u = \underbrace{\frac{\mu_i}{r^{n-2}} + 1 + \sum_{j\neq i}\frac{\mu_j}{|\vec{a}_j - \vec{a}_i|^{n-2}}}_{=:\mathring{u}} + r\hat{u} = \mathring{u}\Big(1 + O(r^{n-1})\Big) , \qquad (1.6.11)$$

with $\hat{u}$ — an analytic function of r and of the angular variables, at least for small r. We choose f so that $\mathring{u}^{\frac{2}{n-2}} - \mathring{u}^{-2}(f')^2$ vanishes:

$$f' = \mathring{u}^{\frac{n-1}{n-2}} .$$

This shows that the function

$$ {}^{n+1}g_{rr} = u^{\frac{2}{n-2}} - u^{-2}(f')^2 = \underbrace{\mathring{u}^{\frac{2}{n-2}}}_{\sim r^{-2}} \left[\left(\frac{u}{\mathring{u}}\right)^{\frac{2}{n-2}} - \left(\frac{\mathring{u}}{u}\right)^2\right] = O(r^{n-3}) $$

is an analytic function of r and angular variables for small r.

The above works well when $n = 3$, in which case (1.6.10) reads

$$ {}^{3+1}g \quad = \quad -\underbrace{u^{-2}}_{\sim r^2} dv^2 + 2\Big(\underbrace{\frac{\mathring{u}}{u}}_{=1+O(r^2)}\Big)^2 dv\,dr + \underbrace{g_{rr}}_{=O(1)} dr^2 + \underbrace{u^2 r^2}_{=\mu_i^2+O(r)} h\ . $$

At $r = 0$ the determinant of ${}^{3+1}g$ equals $-\mu_i^4 \det h \neq 0$, which implies that ${}^{3+1}g_{\mu\nu}$ can be analytically extended across the null hypersurface $\mathscr{H}_i := \{r = 0\}$ to a real-analytic Lorentzian metric defined in a neighborhood of $\mathscr{H}_i$. By analyticity the extended metric is vacuum. Obviously $\mathscr{H}_i$ is a Killing horizon for the Killing vector $\partial_t = \partial_v$, since ${}^{3+1}g_{vv}$ vanishes at $\mathscr{H}_i$.

Let us return to general dimensions $n \geq 4$ and consider ${}^{n+1}g_{rv}$:

$$ \begin{aligned} {}^{n+1}g_{rv}dr\,dv &= u^{-2}f'dr\,dv = \left(\frac{\mathring{u}}{u}\right)^2 \mathring{u}^{\frac{3-n}{n-2}} dr\,dv = \Big(1 + O(r^{n-2})\Big)\mu_i^{\frac{3-n}{n-2}} r^{n-3} dr\,dv \\ &= \Big(1 + O(r^{n-2})\Big)\frac{\mu_i^{\frac{3-n}{n-2}}}{n-2} d(\underbrace{r^{n-2}}_{=:\rho})\,dv \end{aligned} $$

It follows that this term will be better behaved if we introduce a new radial variable $\rho = r^{n-2}$. This, however, will wreak havoc in ${}^{n+1}g_{rr}dr^2$, as well as in various other terms because then $r = \rho^{\frac{1}{n-2}}$, which introduces fractional powers of the new coordinate ρ in the metric. Now, none of these problems occur if $N = 1$, in which case $u = \mathring{u}$, hence ${}^{n+1}g_{rr} \equiv 0$; furthermore,

$$ \begin{aligned} {}^{n+1}g_{vv} = \mathring{u}^{-2} = \Big(1 + \tfrac{\mu_i}{\rho}\Big)^{-2} &= \tfrac{\rho^2}{(\mu_i+\rho)^2}\ , \\ u^{\frac{2}{n-2}} r^2 = (\mathring{u}\rho)^{\frac{2}{n-2}} &= (\mu_i + \rho)^{\frac{2}{n-2}}\ , \\ {}^{n+1}g_{rv}dr\,dv = \tfrac{\mathring{u}^{\frac{3-n}{n-2}}}{(n-2)r^{n-3}} d\rho dv = \tfrac{(\mathring{u}r^{n-2})^{\frac{3-n}{n-2}}}{(n-2)} d\rho dv &= \tfrac{(\mu_i+\rho)^{\frac{3-n}{n-2}}}{(n-2)} d\rho dv\ , \end{aligned} $$

which proves that the metric can be extended analytically across a Killing horizon $\{\rho = 0\}$, as desired. (The case $N = 1$ is of course spherically symmetric, so this calculation is actually a special case of that in Remark 1.2.1.)

For $n \geq 4$ and $N > 1$ the above construction (or some slight variation thereof, with f not necessarily radial, chosen to obtain ${}^{n+1}g_{rr} = 0$) produces a metric which can at best be extended by continuity across a Killing horizon "located at $\vec{x} = \vec{a}_i$", but the extensions so obtained do not appear to be differentiable. The optimal degree of differentiability that one can obtain does not seem to be known; in any case, it has been shown in [75] that the metric cannot be extended smoothly when $n \geq 4$ and $N = 2$ or 3. Similarly, in [5] it is shown that axi-symmetric configurations in $n \geq 5$ do not poseess C^2 extensions.

Problem 1.6.1 Study, for $n \geq 4$, whether (1.6.7) can be corrected by a harmonic function to give a smooth event horizon. Alternatively, show that there are no regular static multi-component electro-vacuum black holes in higher dimensions.

1.7 Emparan-Reall "black rings"

An interesting class of black hole solutions of the 4 + 1 dimensional stationary vacuum Einstein equations has been found by Emparan and Reall [19]. The metrics are asymptotically Minkowskian in spacelike directions, with an ergosurface and an event horizon having $S^1 \times S^2$ cross-sections. (The "ring" terminology refers to the S^1 factor in $S^1 \times S^2$.) Our presentation is an expanded version of [19], with a somewhat different labeling of the contants appearing in the metric; furthermore, the gravitational coupling constant G from that reference has been set to one here.[8]

The starting point of the analysis is the following metric, solution of vacuum Einstein equations:

$$\begin{aligned} g &= -\frac{F(x)}{F(y)}\left(dt + \sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - y}{A}d\psi\right)^2 \\ &\quad + \frac{F(y)}{A^2(x-y)^2}\left[-F(x)\left(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)}d\psi^2\right)\right. \\ &\quad \left. + F(y)\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\right], \end{aligned} \tag{1.7.1}$$

where $A > 0$, ν, and ξ_F are constants, and

$$F(\xi) = 1 - \frac{\xi}{\xi_F}, \tag{1.7.2}$$

$$G(\xi) = \nu\xi^3 - \xi^2 + 1 = \nu(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3). \tag{1.7.3}$$

The constant ν is chosen to satisfy $0 < \nu \leq \nu_* = 2/3\sqrt{3}$. The upper bound is determined by the requirement that the three roots $\xi_1 < \xi_2 < \xi_3$ of G are real. Note that $G(0) = 1$ so that $\xi_1 < 0$. Further $G' = 3\nu\xi^2 - 2\xi > 0$ for $\xi < 0$, which implies that $\xi_2 > 0$. Hence,

$$\xi_1 < 0 < \xi_2 < \xi_3 \ .$$

In our analysis we assume that[9]

$$\xi_2 < \xi_F < \xi_3 \ .$$

a definite choice of ξ_F consistent with this hypothesis will be made shortly.

Requiring that

$$\xi_1 \leq x \leq \xi_2 \tag{1.7.4}$$

guarantees $G(x) \geq 0$ and $F(x) > 0$. On the other hand, both $G(y)$ and $F(y)$ will be allowed to change sign, as we will be working in the ranges

$$y \in (-\infty, \xi_1] \cup (\xi_F, \infty) \ . \tag{1.7.5}$$

[8] I am grateful to R. Emparan and H. Reall for allowing me to reproduce their figures.

[9] According to [19] [19], the choice $\xi_F = \xi_2$ corresponds to the five-dimensional rotating black hole of [52], with one angular momentum parameter set to zero.

Explicit formulae for the roots of G can be found, which are not particularly enlightening. For example, for $\nu \geq \nu_*$ one of the roots reads

$$\frac{\alpha}{6\nu} + \frac{2}{3\nu\alpha} + \frac{1}{3\nu} \,, \quad \text{where } \alpha = \sqrt[3]{-108\,\nu^2 + 8 + 12\,\sqrt{3}\sqrt{27\,\nu^2 - 4}\nu} \,,$$

and a proper understanding of the various roots appearing in this equation also gives all solutions for $0 \leq \nu < \nu_*$. Alternatively, in this last range of ν the roots belong to the set $\{(z_k + \frac{1}{2})\frac{2}{3\nu}\}_{k=0}^2$, with

$$z_k = \cos\left(\frac{1}{3}\left[\arccos\left(1 - \frac{27\nu^2}{2}\right) + 2k\pi\right]\right) .$$

Performing affine transformations of the coordinates, one can always achieve

$$\xi_1 = -1 \,, \quad \xi_2 = 1 \,,$$

but we will not impose these conditions in the calculations that follow.

There is a potential singularity of the $G^{-1}(x)dx^2 + G(x)F^{-1}(x)d\varphi^2$ terms in the metric at $x = \xi_1$, which can be handled as follows: consider, first, a metric of the form

$$h = \frac{dx^2}{x - x_0} + (x - x_0)f(x)d\varphi^2 \,, \quad f(x_0) \neq 0 \,. \tag{1.7.6}$$

Introducing $\tilde\rho = 2\sqrt{x - x_0}$, $\varphi = \lambda\tilde\varphi$ one obtains

$$h = d\tilde\rho^2 + \frac{\lambda^2 f\left(x_0 + \frac{\tilde\rho^2}{4}\right)}{4}\tilde\rho^2 d\tilde\varphi^2 \,. \tag{1.7.7}$$

This defines a metric which smoothly extends through $\tilde\rho = 0$ (when f is smooth) if and only if $\tilde\varphi$ is periodically identified with period, say, 2π, and

$$\lambda = \frac{2}{\sqrt{f(x_0)}} \,. \tag{1.7.8}$$

In order to see that, suppose that (1.7.8) holds, set $\tilde x^1 = \tilde\rho\cos\tilde\varphi$, $\tilde x^2 = \tilde\rho\sin\tilde\varphi$, we then have

$$\begin{aligned} h &= \underbrace{d\tilde\rho^2 + \tilde\rho^2 d\tilde\varphi^2}_{\delta_{ab}d\tilde x^a d\tilde x^b} + \frac{\lambda^2\left(f\left(x_0 + \frac{\tilde\rho^2}{4}\right) - f(x_0)\right)}{4}\underbrace{\tilde\rho^2 d\tilde\varphi^2}_{\delta_{ab}d\tilde x^a d\tilde x^b - d\tilde\rho^2} \\ &= \delta_{ab}d\tilde x^a d\tilde x^b + \frac{\lambda^2\left(f\left(x_0 + \frac{\tilde\rho^2}{4}\right) - f(x_0)\right)}{4}\left(\delta_{ab}d\tilde x^a d\tilde x^b - \tilde\rho^{-2}\tilde x^a \tilde x^b d\tilde x^a d\tilde x^b\right) . \end{aligned}$$

As f is smooth, there exists a smooth function s such that

$$\frac{\lambda^2\left(f\left(x_0 + \frac{\tilde\rho^2}{4}\right) - f(x_0)\right)}{4} = \tilde\rho^2 s(\tilde\rho^2) \,,$$

so that

$$h = \left[\left(1 + s(\tilde\rho^2)\tilde\rho^2\right)\delta_{ab} + s(\tilde\rho^2)\tilde x^a \tilde x^b\right]d\tilde x^a d\tilde x^b \,, \tag{1.7.9}$$

which is manifestly smooth. This shows sufficiency of (1.7.8).

EXERCICE 1.7.1 Show that (1.7.8) is necessary for a smooth complete metric.

In order to apply the above analysis to the last line of (1.7.1) at $x_0 = \xi_1$ we have

$$\begin{aligned}
\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 &= \\
&= \frac{1}{\nu(x-\xi_2)(x-\xi_3)} \left(\frac{dx^2}{x-\xi_1} + \frac{\nu^2 \xi_F (x-\xi_1)(x-\xi_2)^2(x-\xi_3)^2}{\xi_F - x} d\varphi^2 \right) \\
&= \frac{1}{\nu(x-\xi_2)(x-\xi_3)} \left(d\tilde\rho^2 + \frac{\lambda^2\nu^2 \xi_F (x-\xi_2)^2(x-\xi_3)^2}{4(\xi_F - x)} \tilde\rho^2 d\tilde\varphi^2 \right) , \qquad (1.7.10)
\end{aligned}$$

so that (1.7.8) becomes

$$\lambda = \frac{2\sqrt{\xi_F - \xi_1}}{\nu\sqrt{\xi_F}(\xi_2-\xi_1)(\xi_3-\xi_1)}. \qquad (1.7.11)$$

For further purposes it is convenient to rewrite (1.7.10) as

$$\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 = \frac{1}{H(x)} \left[d\tilde\rho^2 + \left(1 + s(\tilde\rho^2)\tilde\rho^2\right) \tilde\rho^2 d\tilde\varphi^2 \right] , \qquad (1.7.12)$$

for a smooth function s with, of course,

$$H(\xi) = \nu(\xi-\xi_2)(\xi-\xi_3) \, . \qquad (1.7.13)$$

When $\xi_F > \xi_2$ one can repeat this analysis at $x = \xi_2$, obtaining instead

$$\lambda = \frac{2\sqrt{\xi_F - \xi_2}}{\nu\sqrt{\xi_F}(\xi_2-\xi_1)(\xi_3-\xi_2)}. \qquad (1.7.14)$$

Equality of (1.7.11) and (1.7.14) determines ξ_F:

$$\xi_F = \frac{\xi_1\xi_2 - {\xi_3}^2}{\xi_1 - 2\xi_3 + \xi_2} \qquad (1.7.15)$$

(Elementary algebra shows that $\xi_2 < \xi_F < \xi_3$, as desired.) It should be clear that with this choice of ξ_F, for $y \neq \xi_1$, the (x,φ)–part of the metric (1.7.1) is a smooth (in fact, analytic) metric on S^2, with the coordinate x being the equivalent of the usual polar coordinate θ on S^2, except possibly at those points where the overall conformal factor vanishes or acquires zeros, which will be analysed shortly. Anticipating, the set obtained by varying x and ϕ and keeping $y = \xi_1$ will be viewed as S^2 with the north pole $x = \xi_1$ removed.

The calculation of the determinant of (1.7.1) reduces to that of a two-by-two determinant in the (t,ψ) variables, which equals

$$\frac{F^2(x)G(y)}{A^2(x-y)^2F(y)} \, , \qquad (1.7.16)$$

leading to

$$\det g = -\frac{F^2(x)F^4(y)}{A^8(x-y)^8} \, , \qquad (1.7.17)$$

so the signature is either $(-++++)$ or $(---++)$, except perhaps at the singular points $x = y$, or $F(x) = 0$ (which does not happen when $\xi_F > \xi_2$, compare (1.7.4)), or $F(y) = 0$.

Now, (1.7.1) is a sum of squares, and $F(x) > 0$, $G(x) > 0$ (away from the axes $x \in \{\xi_1, \xi_2\}$) thus the signature is

$$\Big(\mathrm{sign}(-F(y)), \mathrm{sign}(-G(y)), \mathrm{sign}(-F(y)G(y)), +, +\Big) . \tag{1.7.18}$$

An examination of the four possible cases shows that a Lorentzian signature is obtained except if $F(y) > 0$ and $G(y) > 0$.

We start by considering

$$y \le \xi_1 , \tag{1.7.19}$$

which leads to $F(y) > 0$ and $G(y) \le 0$. Note that $G(\xi_1)$ vanishes; however, it should be clear from what has been said that $-(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)} d\psi^2)$ is a smooth Riemannian metric if $\xi_1 - y$ is related to a radial variable $\hat\rho = 2\sqrt{\xi_1 - y} \in \mathbb{R}^+$ and $\psi = \lambda \hat\varphi$, with λ given by (1.7.14) and $\hat\varphi$ being 2π-periodic. Analogously to (1.7.12), we thus have

$$-\left(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)} d\psi^2\right) = \frac{1}{H(y)}\left[d\hat\rho^2 + \Big(1 + s(\hat\rho^2)\hat\rho^2\Big)\hat\rho^2 d\hat\varphi^2\right] . \tag{1.7.20}$$

Note that the remaining terms in (1.7.1) involving $d\psi$ are also well behaved: indeed, if we set $\hat x^1 = \hat\rho\cos\hat\varphi$, $\hat x^2 = \hat\rho \sin\hat\varphi$, then

$$(\xi_1 - y)d\psi = \frac{\lambda\hat\rho^2}{4} d\hat\varphi = \frac{\lambda}{4}(\hat x^1 d\hat x^2 - \hat x^2 d\hat x^1) ,$$

which is again manifestly smooth.

We turn our attention now to the singularity $x = y$. Given our ranges of coordinates, this only occurs for $x = y = \xi_1$. So, at this stage, the coordinate t parameterises $\mathbb{R}$, the coordinates (y, ψ) are (related to polar) coordinates on $\mathbb{R}^2$, the coordinates (x, φ) are coordinates on S^2. If we think of $x = \xi_1$ as being the north pole of S^2, and we denote it by N, then g is an analytic metric on

$$\underbrace{\mathbb{R}}_{t} \times \Big((\underbrace{\mathbb{R}^2}_{y,\psi \Leftrightarrow \hat\rho,\hat\varphi} \times \underbrace{S^2}_{x,\varphi \Leftrightarrow \tilde\rho,\tilde\varphi}) \setminus (\{0\} \times \{N\}) \Big) .$$

Near the singular set $\mathbb{R} \times \{0\} \times \{N\}$, Emparan and Reall replace $(\tilde\rho, \hat\rho)$ by new radial variables $(\tilde r, \hat r)$ defined as

$$\tilde r = \frac{\tilde\rho}{B(\tilde\rho^2 + \hat\rho^2)} , \quad \hat r = \frac{\hat\rho}{B(\tilde\rho^2 + \hat\rho^2)} , \tag{1.7.21}$$

where B is a constant which will be determined shortly. This is inverted as

$$\tilde\rho = \frac{\tilde r}{B(\tilde r^2 + \hat r^2)} , \quad \hat\rho = \frac{\hat r}{B(\tilde r^2 + \hat r^2)} . \tag{1.7.22}$$

It is convenient to set

$$r = \sqrt{\tilde r^2 + \hat r^2} .$$

We note

$$x = \xi_1 + \frac{\tilde\rho^2}{4} = \xi_1 + \frac{\tilde r^2}{4B^2r^4}\,, \quad y = \xi_1 - \frac{\hat\rho^2}{4} = \xi_1 - \frac{\hat r^2}{4B^2r^4}\,,$$
$$x - y = \frac{1}{4B^2r^2}\,.$$

This last equation shows that $x - y \to 0$ corresponds to region $r \to \infty$.

Inserting (1.7.12) and (1.7.20) into (1.7.1) we obtain

$$\begin{aligned} g &= -\frac{F(x)}{F(y)}\Big(dt + \sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - y}{A}d\psi\Big)^2 \\ &\quad + \frac{F(y)}{A^2(x-y)^2H(x)H(y)}\Bigg[F(x)H(x)\Big(d\hat\rho^2 + (1 + s(\hat\rho^2)\hat\rho^2)\hat\rho^2 d\hat\varphi^2\Big) \\ &\quad + F(y)H(y)\Big(d\tilde\rho^2 + (1+s(\tilde\rho^2)\tilde\rho^2)\tilde\rho^2 d\tilde\varphi^2\Big)\Bigg]\,. \end{aligned} \tag{1.7.23}$$

The simplest terms arise from the first line above:

$$\begin{aligned} &-\frac{\xi_F - \xi_1 - \frac{\tilde r^2}{4B^2r^4}}{\xi_F - \xi_1 + \frac{\hat r^2}{4B^2r^4}}\Big(dt + \sqrt{\frac{\lambda\nu}{\xi_F}}\frac{1}{4AB^2r^4}\hat r^2 d\hat\varphi\Big)^2 \\ &\qquad = -\Big(1 - \frac{1}{4(\xi_F - \xi_1)B^2r^2} + O(r^{-4})\Big)\Big(dt + O(r^{-4})\hat r^2 d\hat\varphi\Big)^2\,. \end{aligned} \tag{1.7.24}$$

In order to analyse the remaining terms, one needs to carefully keep track of all potentially singular terms in the metric: in particular, one needs to make sure that the decay of the metric to the flat one is uniform, including neighborhoods of the rotation axes $\hat r = 0$ and $\tilde r = 0$. So we write

$$\begin{aligned} g_{\hat\varphi\hat\varphi}d\hat\varphi^2 + g_{\tilde\varphi\tilde\varphi}d\tilde\varphi^2 &= \frac{F(y)}{A^2(x-y)^2H(x)H(y)}\Bigg[F(x)H(x)\Big(1 + s(\hat\rho^2)\hat\rho^2\Big)\hat\rho^2 d\hat\varphi^2 \\ &\quad + F(y)H(y)\Big(1 + s(\tilde\rho^2)\tilde\rho^2\Big)\tilde\rho^2 d\tilde\varphi^2\Bigg] \\ &= \frac{4B^2F(y)}{A^2H(x)H(y)}\Bigg[F(x)H(x)\Big(1 + O(r^{-4})\hat r^2\Big)\hat r^2 d\hat\varphi^2 \\ &\quad + F(y)H(y)\Big(1 + O(r^{-4})\tilde r^2\Big)\tilde r^2 d\tilde\varphi^2\Bigg]\,. \end{aligned} \tag{1.7.25}$$

From

$$d\tilde\rho = \frac{1}{Br^4}\Big((\hat r^2 - \tilde r^2)d\tilde r - 2\tilde r\hat r d\hat r\Big)\,, \quad d\hat\rho = \frac{1}{Br^4}\Big((\tilde r^2 - \hat r^2)d\hat r - 2\tilde r\hat r d\tilde r\Big)\,,$$

one finds

$$g_{\hat r\hat r} = \frac{(4B)^2F(y)}{A^2H(x)H(y)r^4}\Big(F(x)H(x)(\hat r^2 - \tilde r^2)^2 + 4F(y)H(y)\hat r^2\tilde r^2\Big)$$

$$
\begin{aligned}
&= \frac{(4B)^2 F(y)}{A^2 H(x) H(y)} \Big(F(x)H(x) + 4(F(y)H(y) - F(x)H(x)) \frac{\hat{r}^2 \tilde{r}^2}{r^4} \Big) \\
&= \frac{(4B)^2 F(y)}{A^2 H(x) H(y)} \Big(F(x)H(x) + O(r^{-4}) \hat{r}^2 \Big) , \qquad (1.7.26) \\
g_{\tilde{r}\tilde{r}} &= \frac{(4B)^2 F(y)}{A^2 H(x) H(y) r^4} \Big(F(y)H(y)(\hat{r}^2 - \tilde{r}^2)^2 + 4F(x)H(x)\hat{r}^2 \tilde{r}^2 \Big) \\
&= \frac{(4B)^2 F(y)}{A^2 H(x) H(y)} \Big(F(y)H(y) + O(r^{-4}) \tilde{r}^2 \Big) , \qquad (1.7.27) \\
g_{\tilde{r}\hat{r}} &= \frac{2(4B)^2 F(y)}{A^2 H(x) H(y) r^4} \hat{r}\tilde{r}(\tilde{r}^2 - \hat{r}^2)(F(y)H(y) - F(x)H(x)) \\
&= O(r^{-4}) \hat{r}\tilde{r} . \qquad (1.7.28)
\end{aligned}
$$

It is clearly convenient to choose B so that

$$
\frac{(4B)^2 F^2(\xi_1)}{A^2 H(\xi_1)} = 1 ,
$$

and with this choice (1.7.24)-(1.7.28) give

$$
\begin{aligned}
g &= -\Big(1 + O(r^{-2})\Big)\Big(dt + O(r^{-4})\hat{r}^2 d\hat{\varphi}\Big)^2 + O(r^{-4})\, \tilde{r} d\tilde{r}\, \hat{r} d\hat{r} \\
&\quad + \Big(1 + O(r^{-2})\Big)\Big(d\hat{r}^2 + \hat{r}^2 d\hat{\varphi}^2\Big) + O(r^{-4}) \hat{r}^4 d\hat{\varphi}^2 \\
&\quad + \Big(1 + O(r^{-2})\Big)\Big(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\varphi}^2\Big) + O(r^{-4}) \tilde{r}^4 d\tilde{\varphi}^2 . \qquad (1.7.29)
\end{aligned}
$$

To obtain a manifestly asymptotically flat form one sets

$$
\hat{y}^1 = \hat{r} \cos \hat{\varphi} \ , \ \hat{y}^2 = \hat{r} \sin \hat{\varphi} \ , \quad \tilde{y}^1 = \tilde{r} \cos \tilde{\varphi} \ , \ \tilde{y}^2 = \tilde{r} \sin \tilde{\varphi} \ ,
$$

then

$$
\begin{aligned}
\hat{r} d\hat{r} &= \hat{y}^1 d\hat{y}^1 + \hat{y}^2 d\hat{y}^2 \ , \quad \hat{r}^2 d\hat{\varphi} = \hat{y}^1 d\hat{y}^2 - \hat{y}^2 d\hat{y}^1 \ , \\
\tilde{r} d\tilde{r} &= \tilde{y}^1 d\tilde{y}^1 + \tilde{y}^2 d\tilde{y}^2 \ , \quad \tilde{r}^2 d\tilde{\varphi} = \tilde{y}^1 d\tilde{y}^2 - \tilde{y}^2 d\tilde{y}^1 \ ,
\end{aligned}
$$

Introducing $(x^\mu) = (t, \hat{y}^1, \hat{y}^2, \tilde{y}^1, \tilde{y}^2)$, (1.7.29) gives a manifestly asymptotically flat metric:

$$
g = \Big(\eta_{\mu\nu} + O(r^{-2})\Big) dx^\mu dx^\nu \ .
$$

In order to understand the geometry when $y \to -\infty$, one replaces y by

$$
Y = -1/y \ .
$$

Surprisingly, the metric can be analytically extended across $\{Y = 0\}$ to negative Y: indeed, we have

$$
\begin{aligned}
g &= -F(x)\Big[\frac{dt^2}{F(y)} + 2\sqrt{\frac{\nu}{\xi_F}} \frac{\xi_1 - y}{AF(y)} dt d\psi \\
&\quad + \frac{1}{A^2}\Big(\frac{\nu(\xi_1 - y)^2}{\xi_F - y} + \frac{G(y)}{(x-y)^2} \Big) d\psi^2 + \frac{F(y) y^4}{A^2 (x-y)^2 G(y)} dY^2 \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{F^2(y)}{A^2(x-y)^2}\left(\frac{dx^2}{G(x)}+\frac{G(x)}{F(x)}d\varphi^2\right)\Bigg]\\
&\qquad\longrightarrow_{y\to-\infty}\\
&-F(x)\left[2\frac{\sqrt{\nu\xi_F}}{A}dtd\psi+\frac{2\xi_1+x-1-\nu\xi_F}{A^2}d\psi^2+\frac{1}{A^2\nu\xi_F}dY^2\right]\\
&+\frac{1}{A^2\xi_F^2}\left(\frac{dx^2}{G(x)}+\frac{G(x)}{F(x)}d\varphi^2\right)\Bigg]\ , \qquad (1.7.30)
\end{aligned}
$$

Calculating directly, or using (1.7.17) and the transformation law for $\det g$, one has

$$\det g = -\frac{F^2(x)F^4(y)y^4}{A^8(x-y)^8} \longrightarrow_{y\to-\infty} -\frac{F^2(x)}{A^8}\ , \qquad (1.7.31)$$

which shows that the metric remains non-degenerate up to $\{Y=0\}$. Further, one checks that all functions in (1.7.30) extend analytically to small negative Y; *e.g.*,

$$g(\partial_t,\partial_t)=g_{tt}=-\frac{F(x)}{F(y)}=-\frac{\xi_F-x}{\xi_F-y}=-\frac{(\xi_F-x)Y}{Y\xi_F+1}\ , \qquad (1.7.32)$$

etc.

To take advantage of the work done so far, in the region $Y<0$ we replace Y by a new coordinate

$$z=-Y^{-1}>0\ ,$$

obtaining a metric which has the same form as (1.7.1):

$$
\begin{aligned}
g \;=\; & -\frac{F(x)}{F(z)}\left(dt+\sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1-z}{A}d\psi\right)^2\\
&+\frac{F(z)}{A^2(x-z)^2}\left[-F(x)\left(\frac{dz^2}{G(z)}+\frac{G(z)}{F(z)}d\psi^2\right)\right.\\
&\left.+F(z)\left(\frac{dx^2}{G(x)}+\frac{G(x)}{F(x)}d\varphi^2\right)\right]\ . \qquad (1.7.33)
\end{aligned}
$$

By continuity, or by (1.7.18), the signature remains Lorentzian, and (taking into account our previous analysis of the zeros of $G(x)$) the metric is manifestly regular in the range

$$\xi_3<z<\infty\ . \qquad (1.7.34)$$

Note, however, that the "stationary" Killing vector ∂_t, which was timelike in the region $Y>0$, is now spacelike in view of (1.7.32). Therefore the region (1.7.34) is an *ergoregion* for the extended space-time. The *ergosurface* at $Y=0$ has topology $S^1\times S^2$ for $\xi_F>\xi_2$, as assumed here.

The above coordinates break down at $z=\xi_3$, so one replaces ψ by a new (periodic) coordinate χ, and t by a new coordinate v, defined as

$$
\begin{aligned}
d\chi &= d\psi+\tfrac{\sqrt{-F(z)}}{G(z)}dz\ ,\\
dv &= dt+\sqrt{\tfrac{\nu}{\xi_F}}(z-\xi_1)\tfrac{\sqrt{-F(z)}}{AG(z)}dz\ .
\end{aligned}
$$

In the new coordinates the metric takes the form

$$\begin{aligned} ds^2 &= -\frac{F(x)}{F(z)}\left(dv - \sqrt{\frac{\nu}{\xi_F}}\frac{z-\xi_1}{A}d\chi\right)^2 \\ &+ \frac{1}{A^2(x-z)^2}\left[F(x)\left(-G(z)d\chi^2 + 2\sqrt{-F(z)}d\chi dz\right)\right. \\ &+ \left. F(z)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right)\right]. \end{aligned} \tag{1.7.35}$$

This is regular at

$$\mathscr{E} := \{z = \xi_3\} ,$$

and the metric can be analytically continued into the region $\xi_F < z \le \xi_3$. One can check directly from (1.7.35) that $g(\nabla z, \nabla z)$ vanishes at $\mathscr{E}$. However, it is simplest to use (1.7.33) to obtain

$$g(\nabla z, \nabla z) = g^{zz} = -\frac{A^2(x-z)^2 G(z)}{F(x)F(z)} \tag{1.7.36}$$

in the region $\{z > \xi_3\}$, and to invoke analyticity to conclude that this equation remains valid on $\{z > \xi_F\}$. Equation (1.7.36) shows that $\mathscr{E}$ is a null hypersurface, with z being a time function on $\{z < \xi_3\}$, which is thus a black hole region by the usual arguments (compare the paragraph around (1.5.6)).

We wish to show that $\{z = \xi_3\}$ is the event horizon: this will follow if we show that there is no event horizon enclosing the region$z < \xi_3$. For this, consider the "area function", defined as the determinant, say W, of the matrix

$$g(K_i, K_j) ,$$

where the K_i's, $i = 1, 2, 3$, are the Killing vectors equal to ∂_t, ∂_ψ, and ∂_φ in the asymptotically flat region. In the original coordinates of (1.7.1) this equals

$$\frac{F(x)G(x)F(y)G(y)}{A^4(x-y)^4} , \tag{1.7.37}$$

with an identical expression where z replaces y in the coordinates of (1.7.33). One further checks that this formula is not affected by the introduction of the coordinates of (1.7.35). Now,

$$F(y)G(y) = \nu\xi_F(\xi_F - y)(y - \xi_1)(y - \xi_2)(y - \xi_3) ,$$

and, in view of the range (1.7.4) of the variable x, the sign of (1.7.37) depends only upon the values of y and z. Since $F(y)$ behaves as $-\nu y^4$ for large y, W is negative both for $y < \xi_1$ and for $z > \xi_3$. Hence, at each point p of those two regions the set of vectors in $T_p\mathscr{M}$ spanned by the Killing vectors is timelike. So, suppose for contradiction, that the event horizon $\mathscr{H}$ intersects the region $\{y \in [-\infty, \xi_1) \cup z \in (\xi_3, \infty]\}$. Since $\mathscr{H}$ is a null hypersurface invariant under isometries, every Killing vector is tangent to $\mathscr{H}$. However, at each point at which W is negative there exists a linear combination of the Killing vectors which is timelike. This gives a contradiction because no timelike vector can be tangent to a null hypersurface.

We conclude that $\{z = \xi_3\}$ forms indeed the event horizon, with topology $\mathbb{R} \times S^1 \times S^2$: this is a "rotating black ring".

It follows from (1.7.35) that the Killing vector field

$$\xi = \frac{\partial}{\partial v} + \frac{A\sqrt{\xi_F}}{\sqrt{\nu}(\xi_3 - \xi_1)}\frac{\partial}{\partial \chi} \tag{1.7.38}$$

is light-like at $\mathscr{E}$, which is therefore a *Killing horizon.* Equation (1.7.38) shows that the horizon is rotating, with angular velocity

$$\Omega_H = \frac{A\sqrt{\xi_F}}{\lambda(\xi_3 - \xi_1)\sqrt{\nu}} = \frac{A\sqrt{\nu}\xi_F(\xi_2 - \xi_1)}{2\sqrt{\xi_F - \xi_1}}. \tag{1.7.39}$$

The surface gravity is

$$\kappa = \frac{A\sqrt{\nu}}{2}\frac{\xi_F(\xi_3 - \xi_2)}{\sqrt{\xi_3 - \xi_F}}. \tag{1.7.40}$$

As $\kappa \neq 0$, one can further extend the space-time obtained so far in the usual way to one which contains a bifurcate Killing horizon, and a white hole region.

The plot of Ω_H and κ (as well as some other quantities of geometric interest) in terms of ν can be found in Figure 1.11.

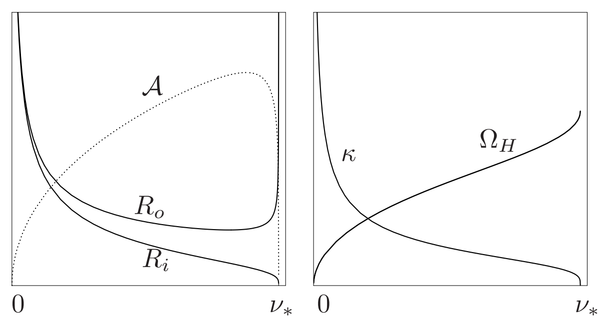

Figure 1.11: Plots, as functions of ν at fixed m, of the radius of curvature R_i at $x = \xi_2$ of the S^1 factor of the horizon, the curvature radius R_o at $x = \xi_1$, total area $\mathcal{A}$ of the ring, surface gravity κ, and angular velocity at the horizon Ω_H. All quantities are rendered dimensionless by dividing by an appropriate power of m. Figure from [19].

It is essential to understand the nature of the orbits of the isometry group, *e.g.* to make sure that the domain of outer communications does not contain any closed timelike curves. We have:

- The Killing vector ∂_t is timelike iff

$$F(y) > 0 \Longleftrightarrow y < \xi_F \ ;$$

- The Killing vector ∂_φ is always spacelike;

- From (1.7.1) we have

$$g(\partial_\psi,\partial_\psi) = \frac{\nu F(x)(\xi_1-y)}{A^2(x-y)^2(\xi_F-y)}\times$$
$$\times \underbrace{\Big((\xi_F-y)(\xi_2-y)(\xi_3-y)-(\xi_1-y)(x-y)^2\Big)}_{(*)} . \quad (1.7.41)$$

For $y<\xi_1$ we can write

$$\underbrace{(\xi_F-y)}_{\geq(x-y)}\underbrace{(\xi_2-y)}_{>(\xi_1-y)}\underbrace{(\xi_3-y)}_{>(x-y)} > (\xi_1-y)(x-y)^2 ,$$

which leads to $g_{\psi\psi}\geq 0$. Similarly, for $y>\xi_3$,

$$\underbrace{(y-\xi_F)}_{\leq(y-x)}\underbrace{(y-\xi_2)}_{<(y-\xi_1)}\underbrace{(y-\xi_3)}_{<(y-x)} < -(\xi_1-y)(x-y)^2 ,$$

so that ∂_ψ is spacelike or vanishing throughout the domain of outer communications.

- The metric induced on the level sets of t has the form

$$g_{yy}dy^2+g_{\psi\psi}d\psi^2+g_{xx}dx^2+g_{\varphi\varphi}d\varphi^2 . \quad (1.7.42)$$

We have just seen that $g_{\psi\psi}$ is non-negative, and g_{xx} and $g_{\varphi\varphi}$ also are in the range (1.7.4). Further

$$g_{yy} = -\frac{F(x)F(y)}{A^2(x-y)^2G(y)} = \frac{F(x)}{A^2(x-y)^2\xi_F\nu}\times\frac{(y-\xi_F)}{(y-\xi_1)(y-\xi_2)(y-\xi_3)} ,$$

an expression which is again positive in the ranges of interest. It follows that the hypersurfaces $\{t=\text{const}\}$ are spacelike.

- The main topological features of the manifold $\mathscr{M}$ constructed so far are summarised in Figure 1.13, see also Figure 1.12. Hence

$$\mathscr{M} = \mathbb{R}\times\Big[\Big(\mathbb{R}^2\times S^2\Big)\setminus\underbrace{(\vec{0},N)}_{=:i^o}\Big] ,$$

where $\vec{0}$ is the origin of $\mathbb{R}^2$, and N is the north pole of S^2, with the first $\mathbb{R}$ factor corresponding to time. The point i^o which has been removed from the $\mathbb{R}^2\times S^2$ factor can be thought of as representing "spatial infinity". It would be of interest to study the maximal analytic extensions of $(\mathscr{M},g)$.

The metric h induced on the sections of the horizon $\{v=\text{const}\,,\ z=\xi_3\}$ can be obtained from (1.7.42) by first neglecting the dy^2 terms, and then passing to the limit $y\to\xi_3$. (By general arguments, or by a direct calculation from (1.7.35), this coincides with the metric of the sections $\{v=\text{const}\}$ of the event horizon $\mathscr{E}$.) One finds

$$h = \frac{\lambda^2\nu(\xi_F-x)(\xi_3-\xi_1)^2}{\xi_F A^2(\xi_3-\xi_F)}d\hat{\varphi}^2 + \frac{F^2(\xi_3)}{A^2(x-\xi_3)^2}\left(\frac{dx^2}{G(x)}+\frac{\lambda^2G(x)}{F(x)}d\tilde{\varphi}^2\right) ,$$

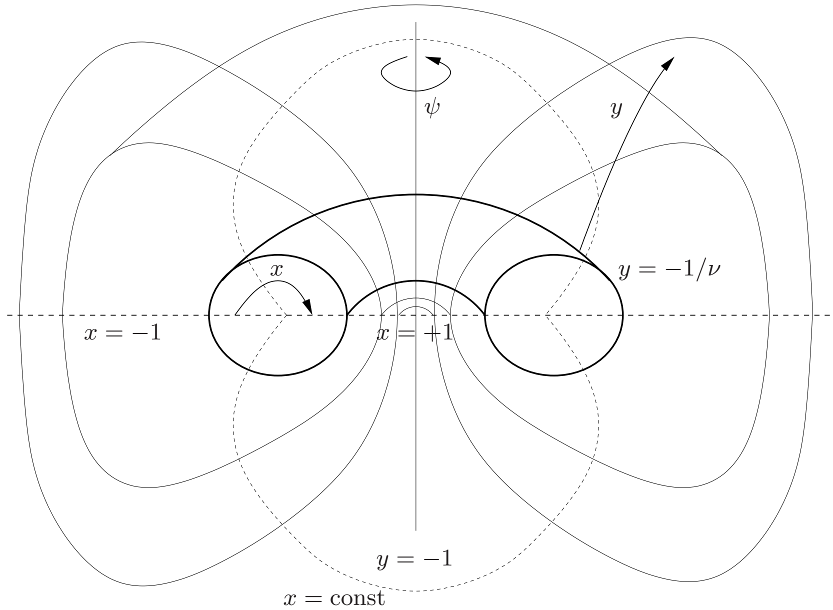

Figure 1.12: Coordinate system for black ring metrics, from [18]. The diagram sketches a section at constant t and φ. Surfaces of constant y are ring-shaped, while x is a polar coordinate on S^2. Infinity lies at $x = y = -1$.

so that (recall (1.7.14))

$$\sqrt{\det h} = \frac{\lambda^2\nu^{1/2}(\xi_3-\xi_F)^{3/2}(\xi_3-\xi_1)}{\xi_F^2 A^3 (x-\xi_3)^2} = \frac{4(\xi_3-\xi_F)^{3/2}(\xi_F-\xi_1)}{A^3\nu^{3/2}\xi_F^3(\xi_3-\xi_1)^2(\xi_2-\xi_1)^2}\times\frac{1}{(x-\xi_3)^2}\,.$$

By integration in $x \in (\xi_1, \xi_2)$ and in the angular variables $\tilde{\varphi}, \hat{\varphi} \in (0, 2\pi)$ one obtains the area of the sections of the event horizon:

$$\mathcal{A} = \frac{16\pi^2}{A^3\nu^{3/2}}\frac{(\xi_3-\xi_F)^{3/2}(\xi_F-\xi_1)}{\xi_F^3(\xi_3-\xi_2)(\xi_2-\xi_1)(\xi_3-\xi_1)^2}\,. \tag{1.7.43}$$

If $\nu = \nu_*$ then the black ring and the black hole degenerate to the same solution with $\xi_2 = \xi_F = \xi_3$. This is the $\mu = a^2$ limit of the five-dimensional rotating black hole, for which the horizon disappears, and is replaced by a naked singularity.

The mass m and the angular momentum J can be calculated using Komar integrals:

$$m = \frac{3\pi}{2A^2}\frac{\xi_F-\xi_1}{\nu\xi_1^2(\xi_2-\xi_1)(\xi_3-\xi_1)}, \tag{1.7.44}$$

$$J = \frac{2\pi}{A^3}\frac{(\xi_F-\xi_1)^{5/2}}{\nu^{3/2}\xi_F^3(\xi_2-\xi_1)^2(\xi_3-\xi_1)^2}. \tag{1.7.45}$$

Thus, m and J are rather complicated functions of the independent parameters A and ν in view of (1.7.15).

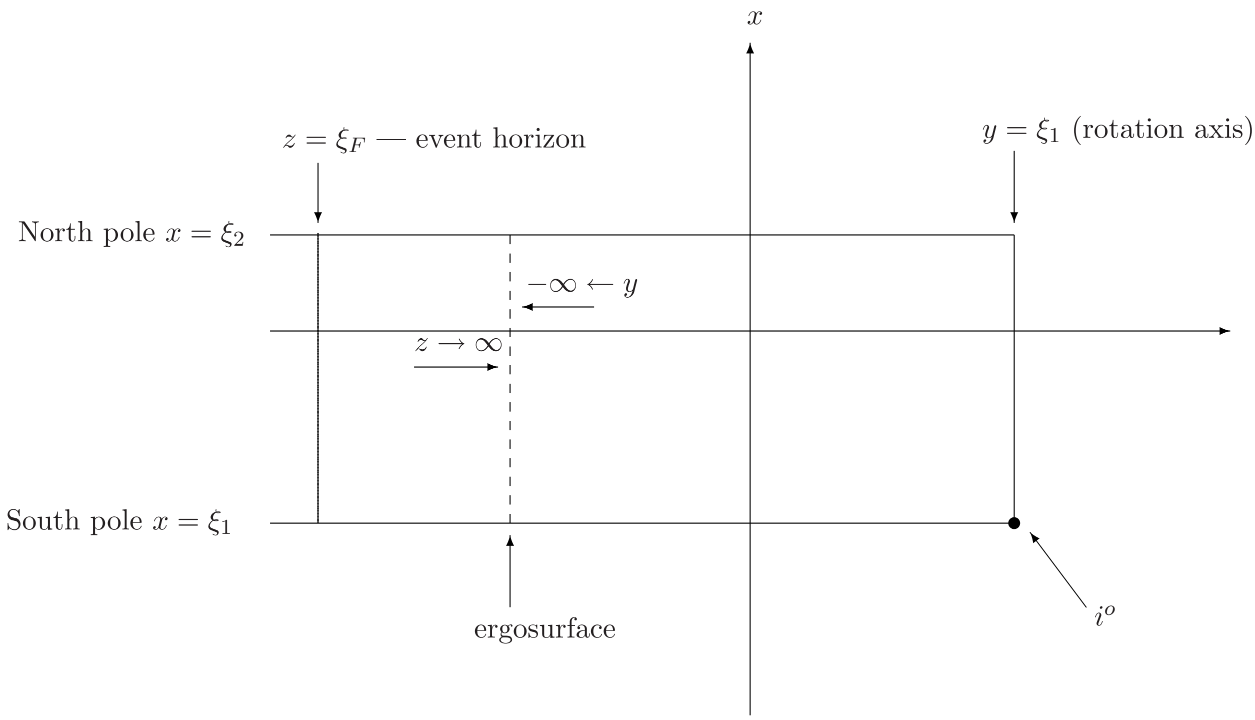

Figure 1.13: Space sections of the Emparan-Reall black holes, with the angular variables φ and ψ suppressed. The x variable runs along the vertical axis, the y variable runs along the horizontal axis to the right of the ergosurface, while the z coordinate is used horizontally to the left of the ergosurface. i^o is the point at infinity.

Recall that the spin of the Myers-Perry five-dimensional black holes is bounded from above [52]:

$$\frac{J^2}{m^3} < \frac{32}{27\pi} \,. \tag{1.7.46}$$

The corresponding ratio for the solutions here is

$$\frac{J^2}{m^3} = \frac{32}{27\pi}\frac{(\xi_3-\xi_1)^3}{(2\xi_3-\xi_1-\xi_2)^2(\xi_2-\xi_1)}. \tag{1.7.47}$$

These ratios are plotted as a function of ν in Figure 1.14. Rather surprisingly, this ratio is bounded *from below*:

$$\frac{J^2}{m^3} > 0.8437\frac{32}{27\pi}. \tag{1.7.48}$$

For $0.2164 < \nu < \nu_*$, there are *two* black ring solutions with the same values of m and J (but different $\mathcal{A}$). Moreover, these satisfy the bound (1.7.46) so there is also a black hole with the same values of m and J. This shows that the uniqueness theorems valid in four dimensions do not have a simple generalisation to five dimensions.

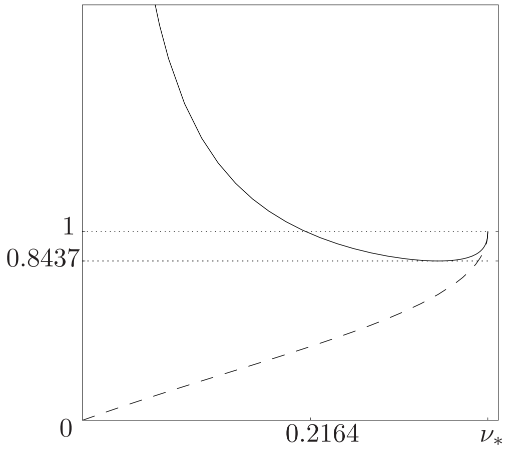

Figure 1.14: $(27\pi/32)J^2/m^3$ as a function of ν. The solid line corresponds to the Emparan-Reall solutions, the dashed line to the Myers-Perry black holes. The two dotted lines delimit the values for which both solutions with the same mass and spin exist. From [19].

Some algebra shows that the quantities m, J, Ω_H, κ and $\mathcal{A}$ satisfy a *Smarr relation*

$$m = \frac{3}{2}\left(\frac{\kappa\mathcal{A}}{8\pi} + \Omega_H J\right). \qquad (1.7.49)$$

Chapter 2

Uniqueness theory

In this chapter we will outline the theory of uniqueness of stationary vacuum black holes, leading to the "no-hair theorems".

The uniqueness proofs of black holes can be divided into two parts: the first is the reduction of the problem to elliptic PDEs, the second is the analysis of those. The reduction involves the orbit-space metric, as defined in (2.1.1) below, and part of the analysis is the understanding of the resulting geometry near Killing horizons. This is the issue that we address in the first section of this chapter.

2.1 The orbit-space geometry near Killing horizons

Consider a spacetime $(\mathscr{M}, \mathfrak{g})$ with a Killing vector field X. On any set $\mathscr{U}$ on which X is timelike we can introduce coordinates in which $X = \partial_t$, and the metric may be written as

$$\mathfrak{g} = -V(dt + \theta_i dx^i)^2 + \gamma_{ij} dx^i dx^j \ , \quad \partial_t V = \partial_t \theta_i = \partial_t \gamma_{ij} = 0 \ . \tag{2.1.1}$$

where $\gamma = \gamma_{ij} dx^i dx^j$ has Riemannian signature. The metric γ is often referred to as the *orbit-space metric*.

In well behaved black-hole spacetimes there usually exists a space-like hypersurface $\mathscr{S} \subset \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, the closure $\overline{\mathscr{S}}$ of which intersects a Killing horizon $\mathscr{N}(X)$ in a compact set; then (2.1.1) defines a Riemannian metric γ on $\mathscr{S} \cap \mathscr{U}$. Assume that X is timelike on $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ near $\mathscr{N}$. The vanishing, or not, of the surface gravity has a deep impact on the geometry of γ near $\mathscr{N}(X)$:

1. Every compact connected component S of $\overline{\mathscr{S}} \cap \mathscr{N}(X)$, included in a C^2 *degenerate* Killing horizon $\mathscr{N}(X)$, on which X does not vanish, corresponds to a *complete* asymptotic end of $(\mathscr{S}, \gamma)$ [13].

2. Every connected component S of $\overline{\mathscr{S}} \cap \mathscr{N}(X)$, included in a smooth Killing horizon $\mathscr{N}(X)$ on which
$$\kappa > 0 \ ,$$
corresponds to a totally geodesic boundary of $(\overline{\mathscr{S}}, \gamma)$, with γ being smooth up–to–boundary. Moreover

(a) a doubling of $(\overline{\mathscr{S}}, \gamma)$ across S leads to a smooth metric on the doubled manifold,

(b) with $\sqrt{-\mathfrak{g}(X,X)}$ extending smoothly to $-\sqrt{-\mathfrak{g}(X,X)}$ across S.

In the Majumdar-Papapetrou solutions of Section 2.5.3, the orbit-space metric γ as in (2.1.1) asymptotes to the usual metric on a round cylinder as the event horizon is approached. One is therefore tempted to think of degenerate event horizons as corresponding to *asymptotically cylindrical ends* of $(\mathscr{S}, \gamma)$.

2.2 Near-horizon geometry

The analysis of the previous section is useful for analysing the elliptic PDEs aspects of the problem at hand. However, to capture the Lorentzian aspects of the problem other tools are neeed. A useful one, to study geometry near smooth null hypersurfaces, is provided by the *null Gaussian coordinates* of Isenberg and Moncrief [**?**]:

Proposition 2.2.1 ([**?**]) *Near a smooth null hypersurface $\mathscr{H}$ one can introduce* Gaussian null coordinates*, in which the space-time metric $\mathfrak{g}$ takes the form*

$$\mathfrak{g} = x\varphi dv^2 + 2dvdx + 2xh_a dx^a dv + h_{ab}dx^a dx^b \; , \tag{2.2.1}$$

with $\mathscr{H}$ given by the equation $\{x = 0\}$.

Proof: Let $S \subset \mathscr{H}$ be any $(n-1)$–dimensional submanifold of $\mathscr{H}$, transverse to the null generators of $\mathscr{H}$. Let x^a be any local coordinate system on S, and let $\ell|_S$ be any field of null vectors, defined on S, tangent to the generators of $\mathscr{H}$. Solving the equation $\nabla_\ell \ell = 0$, with initial values $\ell|_S$ on S, one obtains a null vector field ℓ defined on a $\mathscr{H}$–neighborhood $\mathscr{V} \subset \mathscr{H}$ of S, tangent to the generators of $\mathscr{H}$. One can extend x^a to $\mathscr{V}$ by solving the equation $\ell(x^a) = 0$. The function $v|_{\mathscr{H}}$ is defined by solving the equation $\ell(v) = 1$ with initial value $v|_S = 0$. Passing to a subset of $\mathscr{V}$ if necessary, this defines a global coordinate system (v, x^a) on $\mathscr{V}$. By construction we have $\ell = \partial_v$ on $\mathscr{V}$, in particular $\mathfrak{g}_{vv} = 0$ on $\mathscr{V}$. Further, ℓ is normal to $\mathscr{H}$ because $\mathscr{H}$ is a null surface, which implies $\mathfrak{g}_{va} = 0$ on $\mathscr{V}$.

Let, next, $\bar\ell|_{\mathscr{V}}$ be a field of null vectors on $\mathscr{V}$ defined uniquely by the conditions

$$\mathfrak{g}(\bar\ell|_{\mathscr{V}}, \ell) = 1 \; , \quad \mathfrak{g}(\bar\ell|_{\mathscr{V}}, \partial_A) = 0 \; . \tag{2.2.2}$$

The first equation implies that $\bar\ell|_{\mathscr{V}}$ is everywhere transverse to $\mathscr{V}$. Then we define $\bar\ell$ in a space-time neighborhood $\mathscr{U} \subset \mathscr{M}$ of $\mathscr{V}$ by solving the geodesic equation $\nabla_{\bar\ell}\bar\ell = 0$ with initial value $\bar\ell|_{\mathscr{V}}$ at $\mathscr{V}$. The coordinates (v, x^a) are extended to $\mathscr{U}$ by solving the equations $\bar\ell(v) = \bar\ell(x^a) = 0$, and the coordinate x is defined by solving the equation $\bar\ell(x) = 1$, with initial value $x = 0$ at $\mathscr{V}$. Passing to a subset of $\mathscr{U}$ if necessary, this defines a global coordinate system (v, x, x^a) on $\mathscr{U}$.

By construction we have

$$\bar\ell = \partial_x \; , \tag{2.2.3}$$

hence ∂_x is a null, geodesic, vector field on $\mathscr{U}$. In particular

$$\mathfrak{g}_{xx} \equiv \mathfrak{g}(\partial_x, \partial_x) = 0 .$$

Let $(z^A) = (x, x^a)$, and note that

$$\begin{aligned}\bar{\ell}\Big(\mathfrak{g}(\bar{\ell}, \partial_A)\Big) &= \mathfrak{g}(\bar{\ell}, \nabla_{\bar{\ell}}\partial_A) = \mathfrak{g}(\bar{\ell}, \nabla_{\partial_x}\partial_A) = \mathfrak{g}(\bar{\ell}, \nabla_{\partial_A}\partial_x) \\ &= \mathfrak{g}(\bar{\ell}, \nabla_{\partial_A}\bar{\ell}) = \frac{1}{2}\partial_A\Big(\mathfrak{g}(\bar{\ell}, \bar{\ell})\Big) = 0 .\end{aligned}$$

This shows that the components $\mathfrak{g}_{xA}$ of the metric are x–independent. On S we have $\mathfrak{g}_{xv} = 1$ and $\mathfrak{g}_{xa} = 0$ by (2.2.2), which finishes the proof. □

Example 2.2.2 An example of the coordinate system above is obtained by taking $\mathscr{H}$ to be the light-cone of the origin in $(n+1)$–dimensional Minkowski space-time, with $x = r - t$, $y = (t+r)/2$, then the Minkowski metric η takes the form

$$\eta = -dt^2 + dr^2 + r^2 d\Omega^2 = 2dx\, dy + \frac{(x+2y)^2}{4} d\Omega^2 .$$

By standard causality theory, any null achronal hypersurfaces $\mathscr{H}$ is the union of Lipschitz topological hypersurfaces. Furthermore, through every point $p \in \mathscr{H}$ there is a future inextendible null geodesic entirely contained in $\mathscr{H}$ (though it may leave $\mathscr{H}$ when followed to the past of p). Such geodesics are called *generators*. A topological submanifold S of $\mathscr{H}$ will be called a *local section*, or simply *section*, if S meets the generators of $\mathscr{H}$ transversally; it will be called a *cross-section* if it meets all the generators precisely once.

Let S be any smooth compact cross-section of the horizon, the *average surface gravity* $\langle\kappa\rangle_S$ is defined as

$$\langle\kappa\rangle_S = -\frac{1}{|S|}\int_S \varphi d\mu_h , \tag{2.2.4}$$

where $d\mu_h$ is the measure induced by the metric h on S, and $|S|$ is the volume of S. We emphasise that this is defined regardless of whether or not the stationary Killing vector is tangent to the null generators of the hypersurface.

On a degenerate Killing horizon the surface gravity vanishes, so that the function φ in (2.2.1) can itself be written as xA, for some smooth function A. The vacuum Einstein equations imply (see [**?**, eq. (2.9)] in dimension four and [**?**, eq. (5.9)] in higher dimensions)

$$\mathring{R}_{ab} = \frac{1}{2}\mathring{h}_a\mathring{h}_b - \mathring{D}_{(a}\mathring{h}_{b)} , \tag{2.2.5}$$

where $\mathring{R}_{ab}$ is the Ricci tensor of $\mathring{h}_{ab} := h_{ab}|_{r=0}$, and $\mathring{D}$ is the covariant derivative thereof, while $\mathring{h}_a := h_a|_{r=0}$. The Einstein equations also determine $\mathring{A} := A|_{r=0}$ uniquely in terms of $\mathring{h}_a$ and $\mathring{h}_{ab}$:

$$\mathring{A} = \frac{1}{2}\mathring{h}^{ab}\left(\mathring{h}_a\mathring{h}_b - \mathring{D}_a\mathring{h}_b\right) \tag{2.2.6}$$

(this equation follows again e.g. from [?, eq. (2.9)] in dimension four, and can be checked by a calculation in all higher dimensions). We have:[1] or axial symmetry in space-time dimension four [?]:

THEOREM 2.2.3 ([?]) *Let the space-time dimension be $n+1$, $n \geq 3$, suppose that a degenerate Killing horizon $\mathscr{N}$ has a compact cross-section, and that $\mathring{h}_a = \partial_a \lambda$ for some function λ (which is necessarily the case in vacuum static space-times). Then (2.2.5) implies $\mathring{h}_a \equiv 0$, so that $\mathring{h}_{ab}$ is Ricci-flat.*

THEOREM 2.2.4 ([?]) *In space-time dimension four and in vacuum, suppose that a degenerate Killing horizon $\mathscr{N}$ has a spherical cross-section, and that $(\mathscr{M}, \mathfrak{g})$ admits a second Killing vector field with periodic orbits. For every connected component $\mathscr{N}_0$ of $\mathscr{N}$ there exists an embedding into a Kerr space-time which preserves $\mathring{h}_a$, $\mathring{h}_{ab}$ and $\mathring{A}$.*

It would be of great interest to obtain more information about solutions of (2.2.5), in all dimensions, without any restrictive conditions. For instance, it is expected that the hypothesis of the existence of a second vector field is not necessary for Theorem 2.2.4, and it would of interest to prove, or disprove, this.

In the four-dimensional static case, Theorem 2.2.3 enforces toroidal topology of cross-sections of $\mathscr{N}$, with a flat $\mathring{h}_{ab}$. On the other hand, in the four-dimensional axi-symmetric case, Theorem 2.2.4 guarantees that the geometry tends to a Kerr one, up to second order errors, when the horizon is approached. So, in the degenerate case, the vacuum equations impose strong restrictions on the near-horizon geometry.

It seems that this is not the case any more for non-degenerate horizons, at least in the analytic setting: Indeed, we claim that for any triple $(N, \mathring{h}_a, \mathring{h}_{ab})$, where N is a two-dimensional analytic manifold (compact or not), $\mathring{h}_a$ is an analytic one-form on N, and $\mathring{h}_{ab}$ is an analytic Riemannian metric on N, there exists a vacuum space-time $(\mathscr{M}, \mathfrak{g})$ with a bifurcate (and thus non-degenerate) Killing horizon, so that the metric $\mathfrak{g}$ takes the form (2.2.1) near each Killing horizon branching out of the bifurcation surface $S \approx N$, with $\mathring{h}_{ab} = h_{ab}|_{r=0}$ and $\mathring{h}_a = h_a|_{r=0}$; in fact $\mathring{h}_{ab}$ is the metric induced by $\mathfrak{g}$ on S. When N is the two-dimensional torus $\mathbb{T}^2$ this can be inferred from [?] as follows: using [?, Theorem (2)] with $(\phi, \beta_a, g_{ab})|_{t=0} = (0, 2\mathring{h}_a, \mathring{h}_{ab})$ one obtains a vacuum space-time $(\mathscr{M}' = S^1 \times \mathbb{T}^2 \times (-\epsilon, \epsilon), \mathfrak{g}')$ with a compact Cauchy horizon $S^1 \times \mathbb{T}^2$ and Killing vector K tangent to the S^1 factor of $\mathscr{M}'$. One can then pass to a covering space where S^1 is replaced by $\mathbb{R}$, and use a construction of Rácz and Wald [?, Theorem 4.2] to obtain the desired $\mathscr{M}$ containing the bifurcate horizon. This argument generalises to any analytic $(N, \mathring{h}_a, \mathring{h}_{ab})$ without difficulties.

2.3 Asymptotically flat stationary metrics

There exists several ways of defining asymptotic flatness, all of them roughly equivalent in vacuum. We will adapt a Cauchy data point of view, as it appears to be the least restrictive. So, a space-time $(\mathscr{M}, \mathfrak{g})$ will be said to possess

[1]Some partial results with a non-zero cosmological constant have also been proved in [?].

an *asymptotically flat end* if $\mathscr{M}$ contains a spacelike hypersurface $\mathscr{S}_{\rm ext}$ diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R, with the following properties: there exists a constant $\alpha > 0$ such that, in local coordinates on $\mathscr{S}_{\rm ext}$ obtained from $\mathbb{R}^n \setminus B(R)$, the metric γ induced by $\mathfrak{g}$ on $\mathscr{S}_{\rm ext}$, and the extrinsic curvature tensor K of $\mathscr{S}_{\rm ext}$, satisfy the fall-off conditions, for some $k > 1$,

$$\gamma_{ij} - \delta_{ij} = O_k(r^{-\alpha}) \ , \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}) \ , \tag{2.3.1}$$

where we write $f = O_k(r^\alpha)$ if f satisfies

$$\partial_{k_1} \ldots \partial_{k_\ell} f = O(r^{\alpha-\ell}) \ , \quad 0 \le \ell \le k \ . \tag{2.3.2}$$

For simplicity we assume that the space-time is vacuum, though similar results hold in general under appropriate conditions on matter fields, see [?, ?] and references therein. Along any spacelike hypersurface $\mathscr{S}$, a Killing vector field X of $(\mathscr{M}, \mathfrak{g})$ can be decomposed as

$$X = Nn + Y \ ,$$

where Y is tangent to $\mathscr{S}$, and n is the unit future-directed normal to $\mathscr{S}_{\rm ext}$. The fields N and Y are called "Killing initial data", or KID for short. The vacuum field equations, together with the Killing equations imply the following set of equations on $\mathscr{S}$:

$$D_iY_j + D_jY_i = 2NK_{ij} \ , \tag{2.3.3}$$

$$R_{ij}(\gamma) + K^k{}_kK_{ij} - 2K_{ik}K^k{}_j - N^{-1}(\mathscr{L}_YK_{ij} + D_iD_jN) = 0 \ , \tag{2.3.4}$$

where $R_{ij}(\gamma)$ is the Ricci tensor of γ.

Under the boundary conditions (2.3.1), an analysis of these equations provides detailed information about the asymptotic behavior of (N, Y). In particular one can prove that if the asymptotic region $\mathscr{S}_{\rm ext}$ is contained in a hypersurface $\mathscr{S}$ satisfying the requirements of the positive energy theorem, and if X is timelike along $\mathscr{S}_{\rm ext}$, then $(N, Y^i) \to_{r\to\infty} (A^0, A^i)$, where the A^μ's are constants satisfying $(A^0)^2 > \sum_i (A^i)^2$ [?, ?]. One can then choose adapted coordinates so that the metric can be, locally, written as

$$\mathfrak{g} = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{\gamma_{ij}dx^idx^j}_{=\gamma} \ , \tag{2.3.5}$$

with

$$\partial_t V = \partial_t\theta = \partial_t\gamma = 0 \tag{2.3.6}$$

$$\gamma_{ij} - \delta_{ij} = O_k(r^{-\alpha}) \ , \quad \theta_i = O_k(r^{-\alpha}) \ , \quad V - 1 = O_k(r^{-\alpha}) \ . \tag{2.3.7}$$

As discussed in more detail in [?], in γ-harmonic coordinates, and in e.g. a maximal time-slicing, the vacuum equations for $\mathfrak{g}$ form a quasi-linear elliptic system with diagonal principal part, with principal symbol identical to that of the scalar Laplace operator. Methods known in principle show that, in this "gauge", all metric functions have a full asymptotic expansion in terms of powers of $\ln r$ and inverse powers of r. In the new coordinates we can in fact take

$$\alpha = n - 2 \ . \qquad (2.3.8)$$

By inspection of the equations one can further infer that the leading order corrections in the metric can be written in the Schwarzschild form (1.2.40).

Solutions without $\ln r$ terms are of special interest, because the associated space-times have smooth conformal completion at infinity. In even space-time dimension initial data sets containing such asymptotic regions, when close enough to Minkowskian data, lead to asymptotically simple space-times [**?**, **?**, **?**]. It has been shown by Beig and Simon that logarithmic terms can always be gotten rid of by a change of coordinates in space dimension three when the mass is non-zero [**?**, **?**]. This has been generalised in [**?**] to all stationary metrics in even space-dimension $n \geq 6$, and to static metrics with non-vanishing mass in $n = 5$.

2.4 Domains of outer communications, event horizons

A key notion in the theory of asymptotically flat black holes is that of the *domain of outer communications*, defined as follows: For $t \in \mathbb{R}$ let $\phi_t[X] : \mathscr{M} \to \mathscr{M}$ denote the one-parameter group of diffeomorphisms generated by X; we will write ϕ_t for $\phi_t[X]$ whenever ambiguities are unlikely to occur. Let $\mathscr{S}_{\rm ext}$ be as in Section 2.3, the exterior region $\mathscr{M}_{\rm ext}$ and the *domain of outer communications* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ are then defined as[2]

$$\mathscr{M}_{\rm ext} := \cup_t \phi_t(\mathscr{S}_{\rm ext}) \ , \qquad \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle = I^+(\mathscr{M}_{\rm ext}) \cap I^-(\mathscr{M}_{\rm ext}) \ . \qquad (2.4.1)$$

The *black hole region* $\mathscr{B}$ and the *black hole event horizon* $\mathscr{H}^+$ are defined as

$$\mathscr{B} = \mathscr{M} \setminus I^-(\mathscr{M}_{\rm ext}) \ , \quad \mathscr{H}^+ = \partial \mathscr{B} \ .$$

The *white hole region* $\mathscr{W}$ and the *white hole event horizon* $\mathscr{H}^-$ are defined as above after changing time orientation:

$$\mathscr{W} = \mathscr{M} \setminus I^+(\mathscr{M}_{\rm ext}) \ , \quad \mathscr{H}^- = \partial \mathscr{W} \ .$$

It follows that the boundaries of $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ are included in the event horizons. We set

$$\mathscr{E}^\pm = \partial \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cap I^\pm(\mathscr{M}_{\rm ext}) \ , \qquad \mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^- \ . \qquad (2.4.2)$$

There is considerable freedom in choosing the asymptotic region $\mathscr{S}_{\rm ext}$. However, it is not too difficult to show that $I^\pm(\mathscr{M}_{\rm ext})$, and hence $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$, $\mathscr{H}^\pm$ and $\mathscr{E}^\pm$, are independent of the choice of $\mathscr{S}_{\rm ext}$ as long as the associated $\mathscr{M}_{\rm ext}$'s overlap.

[2]Recall that $I^-(\Omega)$, respectively $J^-(\Omega)$, is the set covered by past-directed timelike, respectively causal, curves originating from Ω, while $\dot I^-$ denotes the boundary of I^-, etc. The sets I^+, etc., are defined as I^-, etc., after changing time-orientation.

2.5 Uniqueness theorems

It is widely expected that the Kerr metrics provide the only stationary, regular, vacuum, four-dimensional black holes. In spite of many works on the subject (see, e.g., [**?**, **?**, 24, **?**, **?**, **?**, **?**, **?**] and references therein), the question is far from being settled.

To describe the current state of affairs, some terminology is needed. A Killing vector X is said to be complete if its orbits are complete, i.e., for every $p \in \mathscr{M}$ the orbit $\phi_t[X](p)$ of X is defined for all $t \in \mathbb{R}$; X is called *stationary* if it is timelike at large distances in the asymptotically flat region. Following [**?**], we introduce the following:

DEFINITION 2.5.1 *Let $(\mathscr{M}, \mathfrak{g})$ be a space-time containing an asymptotically flat end $\mathscr{S}_{\rm ext}$, and let X be stationary Killing vector field on $\mathscr{M}$. We will say that $(\mathscr{M}, \mathfrak{g}, X)$ is I^+*–regular *if X is complete, if the domain of outer communications $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and if $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ contains a spacelike, connected, acausal hypersurface $\mathscr{S} \supset \mathscr{S}_{\rm ext}$, the closure $\overline{\mathscr{S}}$ of which is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotic ends, such that the boundary $\partial\overline{\mathscr{S}} := \overline{\mathscr{S}} \setminus \mathscr{S}$ is a topological manifold satisfying*

$$\partial\overline{\mathscr{S}} \subset \mathscr{E}^+ := \partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \cap I^+(\mathscr{M}_{\rm ext}) \,, \tag{2.5.1}$$

with $\partial\overline{\mathscr{S}}$ meeting every generator of $\mathscr{E}^+$ precisely once.

Some comments about the definition are in order. First one requires completeness of the orbits of the stationary Killing vector because of the need of an action of $\mathbb{R}$ on $\mathscr{M}$ by isometries. Next, one requires global hyperbolicity of the domain of outer communications to guarantee its simple connectedness, to make sure that the *area theorem* [**?**] holds, and to avoid causality violations as well as certain kinds of naked singularities in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Further, the existence of a well-behaved spacelike hypersurface gives reasonable control of the geometry of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, and is a prerequisite to any elliptic PDEs analysis, as is extensively needed for the problem at hand. The existence of compact cross-sections of the future event horizon prevents singularities on the future part of the boundary of the domain of outer communications, e.g. of the kind that occur in the *Curzon solutions* [**?**, **?**], and eventually, together with the area theorem, guarantees the smoothness of that boundary.

Obviously I^+ could have been replaced by I^- throughout the definition, leading to the notion of *I^-–regular* black holes.

The requirement (2.5.1) appears to be somewhat unnatural, as there are perfectly well-behaved hypersurfaces in, e.g., the Schwarzschild space-time which do not satisfy this condition, but there does not seem to be a coherent theory without assuming some version of (2.5.1). The main point of this condition is to avoid certain zeros of the stationary Killing vector X at the boundary of $\mathscr{S}$, which otherwise create various difficulties; e.g.,, it is not clear how to guarantee then smoothness of $\mathscr{E}^+$, or the *"static-or-axisymmetric alternative"*.[3]

Needless to say, I^+–regularity holds for the standard extensions of the solutions of main intereststst: Schwarzschild, Reissner-Nordström, Kerr-Newman, Majumdar-Papapetrou, or Emparan-Reall solutions.

[3] In fact, (2.5.1) is not needed for *static* metric if, e.g., one assumes at the outset that all horizons are non-degenerate, as we do in Theorem 2.5.3 below, see the discussion in the Corrigendum to [13].

We have the following, long-standing conjecture, it being understood that both the Minkowski and the Schwarzschild space-times are members of the Kerr family:

CONJECTURE 2.5.2 *Let $(\mathscr{M},\mathfrak{g})$ be a stationary. vacuum, four-dimensional space-time containing a spacelike, connected, acausal hypersurface $\mathscr{S}$, such that $\overline{\mathscr{S}}$ is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotically flat ends. Suppose that there exists on $\mathscr{M}$ a complete stationary Killing vector X, that $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and that $\partial\overline{\mathscr{S}}\subset\mathscr{M}\setminus\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$. Then $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Kerr space-time.*

2.5.1 Analytic, connected, four dimensional vacuum black holes

The proof of the following, restricted version of Conjecture 2.5.2 can be found in [?]:

THEOREM 2.5.3 *Let $(\mathscr{M},\mathfrak{g})$ be a vacuum, analytic, asymptotically flat, four-dimensional space-time with a stationary Killing vector X such that $(\mathscr{M},\mathfrak{g},X)$ is I^+–regular. If $\mathscr{E}^+$ is connected and mean-non-degenerate, then $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Kerr space-time.*

Theorem 2.5.3 finds its roots in work by Carter and Robinson [?, ?], with further key steps of the proof due to Hawking [?] and Sudarsky and Wald [?]. It should be emphasised that the hypothesis of analyticity and non-degeneracy are highly unsatisfactory, and one believes that they are not needed for the conclusion.One also believes that no solutions with more than one component of $\mathscr{E}^+$ are I^+–regular; this has been established so far only for some special cases [?, ?].

Partial results concerning uniqueness of higher dimensional black holes have been obtained by Hollands and Yazadjiev [?], compare [?, ?, ?, ?, ?].

The proof of Theorem 2.5.3 can be outlined as follows: First, the event horizon in a smooth or analytic space-time is a priori only a Lipschitz surface; so the starting point of the analysis is provided by a result in [?], that event horizons in I^+–regular stationary black hole space-times are as differentiable as the differentiability of the metric allows. One then shows [?][4] that *either* a) the stationary Killing vector is tangent to the generators of the event horizon, *or* b) there exists a second Killing vector defined near the event horizon. The remaining analysis relies heavily on the fact that the domain of outer communications is simply connected [?] (compare [?]).

In case a) one shows that the domain of outer communications contains a maximal (mean curvature zero) spacelike hypersurface [?]; to be able to invoke that last reference one might need, first, to extend $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ using the construction in [?]. This allows one to establish staticity [?], and one concludes using Theorem 2.5.4 below.

[4] Compare [?]; the result, proved by Hawking in space-dimension $n = 3$ [?, ?], has been generalised to $n \geq 4$ by Hollands, Ishibashi and Wald [?].

In case b), analyticity and simple connectedness imply [?] that the isometry group of $(\mathscr{M}, \mathfrak{g})$ contains a $U(1)$ factor, with non-empty axis of rotation. A delicate argument, which finds its roots in the work of Carter [?], proves that the area function

$$W := -\det(\mathfrak{g}(K_a, K_b)) , \qquad a, b = 1, 2$$

where K_a are the stationary and the periodic Killing vector, is strictly positive on the domain of outer communications. As part of the analysis one needs to exclude the possibility that the stationary Killing vector becomes null on the axis of rotation within the domain of outer communications — this is the contents of the *Ergoset theorem*. Classical results on group actions on simply connected manifolds [?, ?] show that the domain of outer communications is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \setminus \overline{B(1)})$, with the action of the isometry group by translations in the first factor, and by rotations around an axis in $\mathbb{R}^3$. The uniformisation theorem allows one to establish that $\sqrt{W}$ can be used as the usual polar coordinate ρ on $\mathbb{R}^3$, leading to a coordinate system in which the field equations reduce to a harmonic map with values in two-dimensional hyperbolic space. The map is singular at the rotation axis (compare [?]), with rather delicate singularity structure at points where the event horizon meets the axis. A uniqueness theorem for such maps [?, ?] achieves the proof.

2.5.2 Static case

Assuming *staticity*, i.e., stationarity and hypersurface-orthogonality of the stationary Killing vector, a much more satisfactory result is available in space dimensions less than or equal to seven, and in higher dimensions on manifolds on which the Riemannian rigid positive energy theorem holds: no analyticity conditions are needed, and non-connected configurations are excluded, without any *a priori* restrictions on the gradient of the norm of the static Killing vector at event horizons.

More precisely, we shall say that a manifold $\widehat{\mathscr{S}}$ is of *positive energy type* if there are no asymptotically flat complete Riemannian metrics on $\widehat{\mathscr{S}}$ with positive scalar curvature and vanishing mass except perhaps for a flat one. This property has been proved so far for all n–dimensional manifolds $\widehat{\mathscr{S}}$ obtained by removing a finite number of points from a compact manifold of dimension $3 \le n \le 7$ [?], or under the hypothesis that $\widehat{\mathscr{S}}$ is spin whatever $n \ge 3$, and is expected to be true in general [?, ?].

We have the following result, which finds its roots in the work of Israel [?], with further simplifications by Robinson [?], and with a significant strengthening by Bunting and Masood-ul-Alam [4]; the proof of the version presented here can be found in [?, 13]:[5]

THEOREM 2.5.4 *Under the hypotheses of Conjecture 2.5.2, suppose moreover that X is hypersurface-orthogonal. Let $\widehat{\mathscr{S}}$ denote the manifold obtained by doubling $\mathscr{S}$ across the non-degenerate components of its boundary and compactifying, in the doubled manifold, all asymptotically flat regions but one to a*

[5] See [?] or the arXiv version of [13] for corrections to some of the claims in [13, ?].

point. If $\widehat{\mathscr{S}}$ is of positive energy type, then $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Schwarzschild space-time.

REMARK 2.5.5 As a corollary of Theorem 2.5.4 one obtains non-existence of static, I^+–regular, vacuum black holes with some components of the horizon degenerate. As observed in [?], if the space-time dimension is four the result follows immediately from Theorem 2.2.3 and from simple connectedness of the domain of outer communications [?], but this does not seem to generalise to higher dimensions in any obvious way.

2.5.3 Multi-black hole solutions

In this section we assume that the space-time dimension is four. Space-times containing several black holes seem to be of particular interest, but we have just seen that, under the conditions spelled-out in Theorem 2.5.4, no such *vacuum* solutions exist in the static class. However, the *Einstein-Maxwell* equations admit static solutions with several black holes: the *Majumdar-Papapetrou* solutions. As already described in Section 2.5.3, the metric $\mathfrak{g}$ and the electromagnetic potential A take the form [43, 59] (compare [14, 23])

$$\mathfrak{g} = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \qquad A = u^{-1}dt\,, \tag{2.5.2}$$

$$u = 1 + \textstyle\sum_{i=1}^I \frac{\mu_i}{|\vec x - \vec a_i|}\,, \tag{2.5.3}$$

for some positive constants μ_i (the electric charges carried by the punctures $\vec x = \vec a_i$). *Standard MP black holes* are obtained if the coordinates x^μ of (1.6.1) cover the range $\mathbb{R}\times(\mathbb{R}^3\setminus\{\vec a_i\})$ for a finite set of points $\vec a_i \in \mathbb{R}^3$, $i = 1, \ldots, I$.

The case $I = 1$ is a special case of the so-called *Reissner-Nordström* metrics, which are the charged, spherically symmetric (connected) generalisations of the Schwarzschild black holes.

The static I^+–regular electro-vacuum black holes are well understood: Indeed, the analysis in [?, 44, 65, 66] (compare [?][5]), leads to:

THEOREM 2.5.6 *Every domain of outer communications in a static, electro-vacuum, black hole space-time satisfying the hypotheses of Conjecture 2.5.2 and which* does not contain degenerate horizons *is isometric to a domain of outer communications of a Reissner-Nordström black hole.*

The relevance of the standard MP black holes follows now from the following result [16]:

THEOREM 2.5.7 *Every domain of outer communications in a static, electro-vacuum, black hole space-time satisfying the hypotheses of Conjecture 2.5.2 and which* contains degenerate horizons *is isometric to a domain of outer communications of a standard MP space-time.*

It thus follows that the MP family provides the only static, electro-vacuum, I^+–regular black holes with *non-connected* horizons.

Part II

Background material

Chapter 3

Introduction to pseudo-Riemannian geometry

3.1 Vector fields

Let M be an n-dimensional manifold. Physicists often think of vector fields in terms of coordinate systems: a vector field X is an object which in a coordinate system $\{x^i\}$ is represented by a collection of functions X^i. In a new coordinate system $\{y^j\}$ the field X is represented by a new set of functions:

$$X^i(x) \to X^j(y) := X^j(x(y))\frac{\partial y^i}{\partial x^j}(x(y)) \ . \tag{3.1.1}$$

(The summation convention is used throughout, so that the index j has to be summed over.)

The notion of a vector field finds its roots in the notion of the tangent to a curve, say $s \to \gamma(s)$. If we use local coordinates to write $\gamma(s)$ as $(\gamma^1(s), \gamma^2(s), \ldots, \gamma^n(s))$, the tangent to that curve at the point $\gamma(s)$ is defined as the set of numbers

$$(\dot\gamma^1(s), \dot\gamma^2(s), \ldots, \dot\gamma^n(s)) \ .$$

Consider, then, a curve $\gamma(s)$ given in a coordinate system x^i and let us perform a change of coordinates $x^i \to y^j(x^i)$. In the new coordinates y^j the curve γ is represented by the functions $y^j(\gamma^i(s))$, with new tangent

$$\frac{dy^j}{ds}(y(\gamma(s))) = \frac{\partial y^j}{\partial x^i}(\gamma(s))\dot\gamma^i(s) \ .$$

This motivates (3.1.1).

In modern differential geometry a different approach is taken: one identifies vector fields with homogeneous first order differential operators acting on real valued functions $f : M \to \mathbb{R}$. In local coordinates $\{x^i\}$ a vector field X will be written as $X^i\partial_i$, where the X^i's are the "physicists's functions" just mentioned. This means that the action of X on functions is given by the formula

$$\boxed{X(f) := X^i\partial_i f} \tag{3.1.2}$$

(recall that ∂_i is the partial derivative with respect to the coordinate x^i). Conversely, given some abstract derivative operator X, the (perhaps locally defined) functions X^i in (3.1.2) can be found by acting on the coordinate functions:

$$X(x^i) = X^i \; . \tag{3.1.3}$$

One justification for the differential operator approach is the fact that the tangent $\dot\gamma$ to a curve γ can be calculated — in a way independent of the coordinate system $\{x^i\}$ chosen to represent γ — using the equation

$$\dot\gamma(f) := \frac{d(f\circ\gamma)}{dt} \; .$$

Indeed, if γ is represented as $\gamma(t) = \{x^i = \gamma^i(t)\}$ within a coordinate patch, then we have

$$\frac{d(f\circ\gamma)(t)}{dt} = \frac{d(f(\gamma(t)))}{dt} = \frac{d\gamma^i(t)}{dt}(\partial_i f)(\gamma(t)) \; ,$$

recovering the previous coordinate formula $\dot\gamma = (d\gamma^i/dt)$. An alternative justification is that this approach does encode the transformation law in a natural way: indeed, from (3.1.3) and (3.1.2) we have

$$X(y^i) = X^j \frac{\partial y^i}{\partial x^j} \; ,$$

reproducing (3.1.1).

At any given point $p \in M$ the set of vectors forms a vector space, denoted by T_pM. The collection of all the tangent spaces is called the tangent bundle to M, denoted by TM.

Covector fields are fields dual to vector fields. It is convenient to define

$$\boxed{dx^i(X) := X^i} \; ,$$

where X^i is as in (3.1.2). With this definition the (locally defined) bases $\{\partial_i\}_{i=1,\ldots,\dim M}$ of TM and $\{dx^j\}_{i=1,\ldots,\dim M}$ of T^*M are dual to each other:

$$\langle dx^i, \partial_j\rangle := dx^i(\partial_j) = \delta^i_j \; ,$$

where δ^i_j is the Kronecker delta, equal to one when $i=j$ and zero otherwise.

Vector fields can be added and multiplied by functions in the obvious way. Another useful operation is the *Lie bracket*, or *commutator*, defined as

$$\boxed{[X,Y](f) := X(Y(f)) - Y(X(f))} \; . \tag{3.1.4}$$

One needs to check that this does indeed define a new vector field: the simplest way is to use local coordinates,

$$\begin{aligned}
[X,Y](f) &= X^j\partial_j(Y^i\partial_i f) - Y^j\partial_j(X^i\partial_i f) \\
&= X^j(\partial_j(Y^i)\partial_i f + Y^i\partial_j\partial_i f) - Y^j(\partial_j(X^i)\partial_i f + X^i\partial_j\partial_i f) \\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f + \underbrace{X^jY^i\partial_j\partial_i f - Y^jX^i\partial_j\partial_i f}_{=X^jY^i\underbrace{(\partial_j\partial_i f - \partial_i\partial_j f)}_{0}} \\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f \; ,
\end{aligned} \tag{3.1.5}$$

which is indeed a homogeneous first order differential operator. Here we have used the symmetry of the matrix of second derivatives of twice differentiable functions. We note that the last line of (3.1.5) also gives an explicit coordinate expression for the commutator of two differentiable vector fields.

Similarly, at any given point $p \in M$ the set of covectors forms a vector space, denoted by T_p^*M. The collection of all the tangent spaces is called the cotangent bundle to M, denoted by T^*M.

3.2 Tensor products

If φ and θ are covectors we can define a bilinear map using the formula

$$(\varphi \otimes \theta)(X,Y) = \varphi(X)\theta(Y) \; . \tag{3.2.1}$$

For example

$$(dx^1 \otimes dx^2)(X,Y) = X^1 Y^2 \; .$$

Using this notation we have

$$g(X,Y) = g(X^i\partial_i, Y^j\partial_j) = \underbrace{g(\partial_j,\partial_j)}_{=:g_{ij}} \underbrace{\underbrace{X^i}_{dx^i(X)} \underbrace{Y^j}_{dx^j(Y)}}_{(dx^i\otimes dx^j(X,Y)} = (g_{ij}dx^i \otimes dx^j)(X,Y)$$

We will write $dx^i dx^j$ for the symmetric product,

$$dx^i dx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i) \; ,$$

and $dx^i \wedge dx^j$ for the anti-symmetric one,

$$dx^i \wedge dx^j := \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i) \; .$$

It should be clear how this generalises: the tensors $dx^i \otimes dx^j \otimes dx^k$, defined as

$$(dx^i \otimes dx^j \otimes dx^k)(X,Y,Z) = X^i Y^j Z^k \; ,$$

form a basis of three-linear maps on the space of vectors, which are objects of the form

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k \; .$$

Here X is a called *tensor of valence* $(0,3)$. Each index transforms as for a covector:

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k = X_{ijk} \frac{\partial x^i}{\partial y^m} \frac{\partial x^j}{\partial y^\ell} \frac{\partial x^k}{\partial y^n} dy^m \otimes dy^\ell \otimes dy^n \; .$$

It is sometimes useful to think of vectors as linear maps on co-vectors, using a formula which looks funny when first met: if θ is a covector, and X is a vector, then

$$X(\theta) := \theta(X) \; .$$

So if $\theta = \theta_i dx^i$ and $X = X^i \partial_i$ then

$$\theta(X) = \theta_i X^i = X^i \theta_i = X(\theta) \ .$$

It then makes sense to define e.g. $\partial_i \otimes \partial_j$ as a bilinear map on covectors:

$$(\partial_i \otimes \partial_j)(\theta, \psi) := \theta_i \psi_j \ .$$

And one can define a map $\partial_i \otimes dx^j$ which is linear on forms in the first slot, and linear in vectors in the second slot as

$$(\partial_i \otimes dx^j)(\theta, X) := \partial_i(\theta) dx^j(X) = \theta_i X^j \ . \tag{3.2.2}$$

The $\partial_i \otimes dx^j$'s form the basis of the space of *tensors of rank* $(1,1)$:

$$T = T^i{}_j \partial_i \otimes dx^j \ .$$

Generally, a *tensor of valence, or rank,* (r,s) can be defined as an object which has r vector indices and s covector indices, so that it transforms as

$$S^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \to S^{m_1 \ldots m_r}{}_{\ell_1 \ldots \ell_s} \frac{\partial y^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial y^{i_s}}{\partial x^{m_r}} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_s}}{\partial y^{j_s}}$$

For example, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$ are vectors, then $X \otimes Y = X^i Y^j \partial_i \otimes \partial_j$ forms a contravariant tensor of valence two.

Tensors of same valence can be added in the obvious way: *e.g.*

$$(A+B)(X,Y) := A(X,Y) + B(X,Y) \quad \Longleftrightarrow \quad (A+B)_{ij} = A_{ij} + B_{ij} \ .$$

Tensors can be multiplied by scalars: *e.g.*

$$(fA)(X,Y,Z) := fA(X,Y,Z) \quad \Longleftrightarrow \quad f(A_{ijk}) := (fA_{ijk}) \ .$$

Finally, we have seen in (3.2.1) how to take tensor products for one forms, and in (3.2.2) how to take a tensor product of a vector and a one form, but this can also be done for higher order tensor; e.g., if S is of valence (a,b) and T is a multilinear map of valence (c,d), then $S \otimes T$ is a multilinear map of valence $(a+c, b+d)$, defined as

$$(S \otimes T)(\underbrace{\theta, \ldots}_{a \text{ covectors and } b \text{ vectors}}, \underbrace{\psi, \ldots}_{c \text{ covectors and } d \text{ vectors}}) := S(\theta, \ldots) T(\psi, \ldots) \ .$$

3.2.1 Contractions

Given a tensor field $S^i{}_j$ with one index down and one index up one can perform the sum

$$S^i{}_i \ .$$

This defines a scalar, i.e., a function on the manifold. Indeed, using the transformation rule

$$S^i{}_j \to \bar{S}^\ell{}_k = S^i{}_j \frac{\partial x^j}{\partial y^k} \frac{\partial y^\ell}{\partial x^i} \ ,$$

one finds

$$\bar{S}^\ell{}_\ell = S^i{}_j \underbrace{\frac{\partial x^j}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^i}}_{\delta^j_i} = S^i{}_i \;,$$

as desired.

One can similarly do contractions on higher valence tensors, e.g.

$$S^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 j_3 \ldots j_s} \to S^{\ell i_2 \ldots i_r}{}_{j_1 \ell j_3 \ldots j_s} \;.$$

After contraction, a tensor of rank $(r+1, s+1)$ becomes of rank (r, s).

3.3 Raising and lowering of indices

Let g be a symmetric two-covariant tensor field on M, by definition such an object is the assignment to each point $p \in M$ of a bilinear map $g(p)$ from $T_pM \times T_pM$ to $\mathbb{R}$, with the additional property

$$g(X, Y) = g(Y, X) \;.$$

In this work the symbol g will be reserved to *non-degenerate* symmetric two-covariant tensor fields. It is usual to simply write g for $g(p)$, the point p being implicitly understood. We will sometimes write g_p for $g(p)$ when referencing p will be useful.

The usual Sylvester's inertia theorem tells us that at each p the map g will have a well defined signature; clearly this signature will be point-independent on a connected manifold when g is non-degenerate. A pair (M, g) is said to be a *Riemannian manifold* when the signature of g is $(\dim M, 0)$; equivalently, when g is a positive definite bilinear form on every product $T_pM \times T_pM$. A pair (M, g) is said to be a *Lorentzian manifold* when the signature of g is $(\dim M - 1, 1)$. One talks about *pseudo-Riemannian* manifolds whatever the signature of g, as long as g is non-degenerate, but we will only encounter Riemannian and Lorentzian metrics in this work.

Since g is non-degenerate it induces an isomorphism

$$\flat : T_pM \to T_p^*M$$

by the formula

$$\boxed{X_\flat(Y) = g(X, Y)} \;.$$

In local coordinates this gives

$$X_\flat = g_{ij} X^i dx^j =: X_j dx^j \;. \tag{3.3.1}$$

This last equality defines X_j — "the vector X^j with the index j lowered":

$$\boxed{X_i := g_{ij} X^j} \;. \tag{3.3.2}$$

The operation (3.3.2) is called the *lowering of indices* in the physics literature and, again in the physics literature, one does not make a distinction between the one-form $X_\flat$ and the vector X.

The inverse map will be denoted by $\sharp$ and is called the *raising of indices*; from (3.3.1) we obviously have

$$\alpha^\sharp = g^{ij}\alpha_i\partial_j =: \alpha^i\partial_i \quad \Longleftrightarrow \quad dx^i(\alpha^\sharp) = \boxed{\alpha^i = g^{ij}\alpha_j}\ ,$$

where g^{ij} is the matrix inverse to g_{ij}. For example,

$$(dx^i)^\sharp = g^{ik}\partial_k \ .$$

Clearly g^{ij}, understood as the matrix of a bilinear form on T_p^*M, has the same signature as g, and can be used to define a scalar product $g^\sharp$ on $T_p^*(M)$:

$$g^\sharp(\alpha,\beta) := g(\alpha^\sharp,\beta^\sharp) \quad \Longleftrightarrow \quad g^\sharp(dx^i,dx^j) = g^{ij} \ .$$

This last equality is justified as follows:

$$g^\sharp(dx^i,dx^j) = g((dx^i)^\sharp,(dx^j)^\sharp) = g(g^{ik}\partial_k, g^{j\ell}\partial_\ell) = \underbrace{g^{ik}g_{k\ell}}_{=\delta^i_\ell}g^{j\ell} = g^{ji} = g^{ij} \ .$$

It is convenient to use the same letter g for $g^\sharp$ — physicists do it all the time — or for scalar products induced by g on all the remaining tensor bundles, and we will sometimes do so.

3.4 Covariant derivatives

When dealing with $\mathbb{R}^n$, or subsets thereof, there exists an obvious prescription how to differentiate tensor fields: we have then at our disposal the canonical trivialization $\{\partial_i\}_{i=1,\ldots,n}$ of $T\mathbb{R}^n$, together with its dual trivialization $\{dx^j\}_{i=1,\ldots,n}$ of $T^*\mathbb{R}^n$. We can expand a tensor field T of valence (k,ℓ) in terms of those bases,

$$\begin{aligned} T &= T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \\ &\Longleftrightarrow \quad T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell} = T(dx^{i_1},\ldots,dx^{i_k},\partial_{j_1},\ldots,\partial_{j_\ell}) \ , \end{aligned} \tag{3.4.1}$$

and differentiate each component $T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}$ of T separately:

$$X(T) := X^i\partial_i(T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell})\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \ . \tag{3.4.2}$$

The resulting object does, however, *not* behave as a tensor under coordinate transformations: as an example, consider the one-form $T = dx$ on $\mathbb{R}^n$, which has vanishing derivative as defined by (3.4.2). When expressed in spherical coordinates we have

$$T = d(\rho\cos\varphi) = -\rho\sin\varphi d\varphi + \cos\varphi d\rho \ ,$$

the partial derivatives of which are non-zero, both with respect to the original cartesian coordinates (x,y) and to the new spherical ones (ρ,φ). The notion of a *covariant derivative*, sometimes also referred to as a *connection*, is introduced precisely to obtain a notion of derivative which has tensorial properties. By definition, a covariant derivative is a map which to a vector field X and a tensor field T assigns a tensor field of the same type as T, denoted by $\nabla_X T$, with the following properties:

1. $\nabla_X T$ is linear with respect to addition both with respect to X and T:
$$\nabla_{X+Y}T = \nabla_X T + \nabla_Y T \,, \qquad \nabla_X(T+Y) = \nabla_X T + \nabla_X Y \,, \tag{3.4.3}$$
2. $\nabla_X T$ is linear with respect to multiplication of X by functions f,
$$\nabla_{fX}T = f\nabla_X T \,, \tag{3.4.4}$$
3. and, finally, $\nabla_X T$ satisfies the *Leibniz rule* under multiplication of T by a differentiable function f:
$$\nabla_X(fT) = f\nabla_X T + X(f)T \,. \tag{3.4.5}$$

It is natural to ask whether covariant derivatives do exist at all in general and, if so, how many of them can there be. First, it immediately follows from the axioms above that if D and ∇ are two covariant derivatives, then
$$\Delta(X,T) := D_X T - \nabla_X T$$
is multi-linear both with respect to addition and multiplication by functions — the non-homogeneous terms $X(f)T$ in (3.4.5) cancel out — and is thus a tensor field. Reciprocally, if D is a covariant derivative and $\Delta(X,T)$ is bilinear with respect to addition and multiplication by functions, then
$$\nabla_X T := D_X T + \Delta(X,T) \tag{3.4.6}$$
is a new covariant derivative.

We note that the sum of two covariant derivatives is *not* a covariant derivative. However, *convex* combinations of covariant derivatives, with coefficients which may vary from point to point, are again covariant derivatives. This remark allows one to construct covariant derivatives using partitions of unity: Let, indeed, $\{\mathscr{O}_i\}_{i\in\mathbb{N}}$ be an open covering of M by coordinate patches and let φ_i be an associated partition of unity. In each of those coordinate patches we can decompose a tensor field T as in (3.4.1), and define
$$D_X T := \sum_i \varphi_i X^j \partial_j (T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell})\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \,. \tag{3.4.7}$$

This procedure, which depends upon the choice of the coordinate patches and the choice of the partition of unity, defines *one* covariant derivative; all other covariant derivatives are then obtained from D using (3.4.6). Note that (3.4.2) is a special case of (3.4.7) when there exists a global coordinate system on M. Thus (3.4.2) *does* define a covariant derivative. However, the associated operation on tensor fields will *not* take the simple form (3.4.2) when we go to a different coordinate system $\{y^i\}$ in general.

As an illustration, let us describe all possible covariant derivatives on functions: first, it is straightforward to check that the assignment
$$(X,f) \longrightarrow X(f) \tag{3.4.8}$$

is a covariant derivative. It then follows that prescribing a covariant derivative on functions is equivalent to prescribing a field ω of one-forms with

$$\nabla_X f = X(f) + \omega(X) f \ . \tag{3.4.9}$$

Clearly, any one-form

$$\omega(X) = \nabla_X 1$$

determines a unique covariant derivative on functions by (3.4.9). We are free to choose ω arbitrarily, and each covariant derivative on functions is uniquely determined by some ω. For functions the generalization obtained by adding a ω piece is not very useful, and throughout this work only the covariant derivative (3.4.8) will be used for functions. The addition of a lower order term in ∇ becomes, however, a necessity when one wishes to construct tensors by differentiation of tensors other than functions.

The simplest next possibility is that of a covariant derivative of vector fields. We will first assume that we are working on a set $\Omega \subset M$ over which we have a *global trivialization* of the tangent bundle TM; by definition, this means that there exist vector fields e_a, $a = 1, \ldots, \dim M$, such that at every point $p \in \Omega$ the fields $e_a(p) \in T_pM$ form a basis of T_pM.[1] Let θ^a denote the dual trivialization of T^*M — by definition the θ^a's satisfy

$$\boxed{\theta^a(e_b) = \delta^a_b} \ .$$

Given a covariant derivative ∇ on vector fields we set

$$\Gamma^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_b(X) e_a \ , \tag{3.4.10a}$$

$$\boxed{\Gamma^a{}_{bc} := \Gamma^a{}_b(e_c) = \theta^a(\nabla_{e_c} e_b)} \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_{bc} X^c e_a \ . \tag{3.4.10b}$$

The (locally) defined) functions $\Gamma^a{}_{bc}$ are called *connection coefficients*. If $\{e_a\}$ is the coordinate basis $\{\partial_\mu\}$ we shall write

$$\Gamma^\mu{}_{\alpha\beta} := dx^\mu(\nabla_{\partial_\beta}\partial_\alpha) \quad \Big(\Longleftrightarrow \quad \nabla_{\partial_\mu}\partial_\nu = \Gamma^\sigma{}_{\nu\mu}\partial_\sigma \Big) \ , \tag{3.4.11}$$

etc. In this particular case the connection coefficients are usually called *Christoffel symbols*. We will sometimes write $\Gamma^\sigma_{\nu\mu}$ instead of $\Gamma^\sigma{}_{\nu\mu}$. By using the Leibniz rule (3.4.5) we find

$$\begin{aligned}
\nabla_X Y &= \nabla_X(Y^a e_a) \\
&= X(Y^a) e_a + Y^a \nabla_X e_a \\
&= X(Y^a) e_a + Y^a \Gamma^b{}_a(X) e_b \\
&= (X(Y^a) + \Gamma^a{}_b(X) Y^b) e_a \\
&= (X(Y^a) + \Gamma^a{}_{bc} Y^b X^c) e_a \ ,
\end{aligned} \tag{3.4.12}$$

[1]This is the case when Ω is a coordinate patch with coordinates (x^i), then the $\{e_a\}_{a=1,\ldots,\dim M}$ can be chosen to be equal to $\{\partial_i\}_{a=1,\ldots,\dim M}$. Recall that a manifold is said to be parallelizable if a basis of TM can be chosen globally over M — in such a case Ω can be taken equal to M. We emphasize that we are *not* assuming that M is parallelizable, so that equations such as (3.4.10) have only a local character in general.

which gives various equivalent ways of writing $\nabla_X Y$. The (perhaps only locally defined) $\Gamma^a{}_b$'s are linear in X, and the collection $(\Gamma^a{}_b)_{a,b=1,\ldots,\dim M}$ is sometimes referred to as the *connection one-form.* The one-covariant, one-contravariant tensor field ∇Y is defined as

$$\nabla Y := \nabla_a Y^b \theta^a \otimes e_b \Longleftrightarrow \nabla_a Y^b := \theta^b(\nabla_{e_a} Y) \Longleftrightarrow \boxed{\nabla_a Y^b = e_a(Y^b) + \Gamma^b{}_{ca} Y^c} \, . \tag{3.4.13}$$

We will sometimes write ∇_a for ∇_{e_a}. Further, $\nabla_a Y^b$ will sometimes be written as $Y^b{}_{;a}$. It should be stressed that the notation $\nabla_a Y^b$ does *not* mean the action of a derivative operator ∇_a on a component Y^b of a vector field (as would have been the case if the Y^a's were treated as functions, as in (3.4.9)), but represents the tensor field ∇Y as in (3.4.13).

Suppose that we are given a covariant derivative on vector fields, there is a natural way of inducing a covariant derivative on one-forms by imposing the condition that *the duality operation be compatible with the Leibniz rule*: given two vector fields X and Y together with a field of one-forms α one sets

$$\boxed{(\nabla_X \alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)} \, . \tag{3.4.14}$$

Let us, first, check that (3.4.14) defines indeed a field of one-forms. The linearity, in the Y variable, with respect to addition is obvious. Next, for any function f we have

$$\begin{aligned} (\nabla_X \alpha)(fY) &= X(\alpha(fY)) - \alpha(\nabla_X (fY)) \\ &= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y + f\nabla_X Y) \\ &= f(\nabla_X \alpha)(Y) \, , \end{aligned}$$

as should be the case for one-forms. Next, we need to check that ∇ defined by (3.4.14) does satisfy the remaining axioms imposed on covariant derivatives. Again multi-linearity with respect to additions is obvious, as well as linearity with respect to multiplication of X by a function. Finally,

$$\begin{aligned} \nabla_X (f\alpha)(Y) &= X(f\alpha(Y)) - f\alpha(\nabla_X Y) \\ &= X(f)\alpha(Y) + f(\nabla_X \alpha)(Y) \, , \end{aligned}$$

as desired.

The duality pairing

$$T^*_p M \times T_p M \ni (\alpha, X) \to \alpha(X) \in \mathbb{R}$$

is sometimes called *contraction.* As already pointed out, the operation ∇ on one forms has been defined in (3.4.14) so as to satisfy the *Leibniz rule under duality pairing*:

$$X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \, ; \tag{3.4.15}$$

this follows directly from (3.4.14). This should not be confused with the Leibniz rule under multiplication by functions, which is part of the definition of a covariant derivative, and therefore always holds. It should be kept in mind that (3.4.15) does not necessarily hold for all covariant derivatives: if ${}^v\nabla$ is some covariant derivative on vectors, and ${}^f\nabla$ is some covariant derivative on one-forms, in general one will have

$$X(\alpha(Y)) \neq ({}^f\nabla_X)\alpha(Y) + \alpha({}^v\nabla_X Y)\ .$$

Using the basis-expression (3.4.12) of $\nabla_X Y$ and the definition (3.4.14) we have

$$\nabla_X \alpha = X^a \nabla_a \alpha_b \, \theta^b\ ,$$

with

$$\begin{aligned} \boxed{\nabla_a \alpha_b} &:= (\nabla_{e_a}\alpha)(e_b) \\ &= e_a(\alpha(e_b)) - \alpha(\nabla_{e_a} e_b) \\ &= \boxed{e_a(\alpha_b) - \Gamma^c{}_{ba}\alpha_c}\ . \end{aligned}$$

It should now be clear how to extend ∇ to tensors of arbitrary valence: if T is r covariant and s contravariant one sets

$$\begin{aligned} (\nabla_X T)(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) &:= X\Big(T(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)\Big) \\ &-T(\nabla_X X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) - \ldots - T(X_1,\ldots,\nabla_X X_r,\alpha_1,\ldots\alpha_s) \\ &-T(X_1,\ldots,X_r,\nabla_X\alpha_1,\ldots\alpha_s) - \ldots - T(X_1,\ldots,X_r,\alpha_1,\ldots\nabla_X\alpha_s)\ . \end{aligned} \tag{3.4.16}$$

The verification that this defines a covariant derivative proceeds in a way identical to that for one-forms. In a basis we have

$$\nabla_X T = X^a \nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s}\theta^{a_1}\otimes\ldots\otimes\theta^{a_r}\otimes e_{b_1}\otimes\ldots\otimes e_{b_s}\ ,$$

and (3.4.16) gives

$$\begin{aligned} \nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s} &:= (\nabla_{e_a}T)(e_{a_1},\ldots,e_{a_r},\theta^{b_1},\ldots,\theta^{b_s}) \\ = \ & e_a(T_{a_1\ldots a_r}{}^{b_1\ldots b_s}) - \Gamma^c{}_{a_1 a}T_{c\ldots a_r}{}^{b_1\ldots b_s} - \ldots - \Gamma^c{}_{a_r a}T_{a_1\ldots c}{}^{b_1\ldots b_s} \\ &+\Gamma^{b_1}{}_{ca}T_{a_1\ldots a_r}{}^{c\ldots b_s} + \ldots + \Gamma^{b_s}{}_{ca}T_{a_1\ldots a_r}{}^{b_1\ldots c}\ . \end{aligned} \tag{3.4.17}$$

Carrying over the last two lines of (3.4.16) to the left-hand-side of that equation one obtains the Leibniz rule for ∇ under pairings of tensors with vectors or forms. It should be clear from (3.4.16) that ∇ defined by that equation is the *only covariant derivative which agrees with the original one on vectors, and which satisfies the Leibniz rule under the pairing operation.* We will only consider such covariant derivatives in this work.

3.4.1 Torsion

Let ∇ be a covariant derivative defined for vector fields, the *torsion tensor* T is defined by the formula

$$\boxed{T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]}\ , \tag{3.4.18}$$

where $[X,Y]$ is the Lie bracket defined in (3.1.4). We obviously have

$$T(X,Y) = -T(Y,X) \; . \tag{3.4.19}$$

Let us check that T is actually a tensor field: multi-linearity with respect to addition is obvious. To check what happens under multiplication by functions, in view of (3.4.19) it is sufficient to do the calculation for the first slot of T. We then have

$$\begin{aligned} T(fX,Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y] \\ &= f\Big(\nabla_X Y - \nabla_Y X\Big) - Y(f)X - [fX,Y] \; . \end{aligned} \tag{3.4.20}$$

To work out the last commutator term we compute, for any function g,

$$[fX,Y](g) = fX(Y(g)) - \underbrace{Y(fX(g))}_{=Y(f)X(g)+fY(X(g))} = f[X,Y](g) - Y(f)X(g) \; ,$$

hence

$$[fX,Y] = f[X,Y] - Y(f)X \; , \tag{3.4.21}$$

and the last term here cancels the undesirable before-last term in (3.4.20), as required.

In a coordinate basis ∂_μ we have $[\partial_\mu,\partial_\nu]=0$ and one finds from (3.4.11)

$$\boxed{T_{\mu\nu} := T(\partial_\mu,\partial_\nu) = (\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu})\partial_\sigma} \; , \tag{3.4.22}$$

which shows that — in coordinate frames — T is determined by twice the antisymmetrization of the $\Gamma^\sigma{}_{\mu\nu}$'s over the lower indices. In particular that last antisymmetrization produces a tensor field.

3.4.2 Transformation law

Consider a coordinate basis ∂_{x^i}, it is natural to enquire about the transformation law of the connection coefficients $\Gamma^i{}_{jk}$ under a change of coordinates $x^i \to y^k(x^i)$. To make things clear, let us write $\Gamma^i{}_{jk}$ for the connection coefficients in the x–coordinates, and $\hat\Gamma^i{}_{jk}$ for the ones in the y–cordinates. We calculate:

$$\begin{aligned} \Gamma^i{}_{jk} &:= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\Big) \\ &= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}\Big) \\ &= dx^i\Big(\frac{\partial^2 y^\ell}{\partial x^k\partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial y^\ell}\Big) \\ &= \frac{\partial x^i}{\partial y^s}dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k\partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial y^r}{\partial x^k}\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\ &= \frac{\partial x^i}{\partial y^s}dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k\partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\ &= \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k\partial x^j} + \frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\hat\Gamma^s{}_{\ell r} \; . \end{aligned} \tag{3.4.23}$$

Summarising,

$$\boxed{\Gamma^i{}_{jk} = \hat\Gamma^s{}_{\ell r}\frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k} + \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j}} \,. \qquad (3.4.24)$$

Thus, the $\Gamma^i{}_{jk}$'s do *not* form a tensor; instead they transform as a tensor *plus* a non-homogeneous second derivatives term above.

3.4.3 The Levi-Civita connection

One of the fundamental results in pseudo-Riemannian geometry is that of existence of a torsion-free connection which preserves the metric:

THEOREM 3.4.1 *Let g be a two-covariant symmetric non-degenerate tensor field on a manifold M. Then there exists a unique connection ∇ such that*

1. $\nabla g = 0$,
2. *the torsion tensor T of ∇ vanishes.*

PROOF: Let us start with uniqueness. Suppose, thus, that a connection satisfying the above is given, by the Leibniz rule we then have for any vector fields X, Y and Z,

$$0 = (\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \,. \qquad (3.4.25)$$

One then rewrites the same equation applying cyclic permutations to X, Y, and Z, with a minus sign for the last equation:

$$\begin{aligned} +g(\nabla_X Y, Z) + g(Y, \nabla_X Z) &= X(g(Y,Z))\,, \\ +g(\nabla_Y Z, X) + g(Z, \nabla_Y X) &= Y(g(Z,X))\,, \\ -g(\nabla_Z X, Y) - g(X, \nabla_Z Y) &= -Z(g(X,Y))\,. \end{aligned} \qquad (3.4.26)$$

As the torsion tensor vanishes, the sum of the left-hand-sides of these equations can be manipulated as follows:

$$\begin{aligned} & g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\ &= g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &= g(2\nabla_X Y - [X,Y], Z) + g(Y, [X,Z]) + g(X, [Y,Z]) \\ &= 2g(\nabla_X Y, Z) - g([X,Y], Z) + g(Y, [X,Z]) + g(X, [Y,Z]) \,. \end{aligned}$$

This shows that the sum of the three equations (3.4.26) can be rewritten as

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g([X,Y],Z) - g(Y,[X,Z]) - g(X,[Y,Z]) \\ &\quad + X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \,. \end{aligned} \qquad (3.4.27)$$

Since Z is arbitrary and g is non-degenerate, the left-hand-side of this equation determines the vector field $\nabla_X Y$ uniquely, and uniqueness of ∇ follows.

To prove existence, let $S(X,Y)(Z)$ be defined as one half of the right-hand-side of (3.4.27),

$$\begin{aligned} S(X,Y)(Z) &= \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g(Z,[X,Y]) - g(Y,[X,Z]) - g(X,[Y,Z])\Big) . \end{aligned} \tag{3.4.28}$$

Clearly S is linear with respect to addition in all fields involved. It is straightforward to check that it is linear with respect to multiplication of Z by a function, and since g is non-degenerate there exists a unique vector field $W(X,Y)$ such that

$$S(X,Y)(Z) = g(W(X,Y),Z) .$$

One readily checks that the assignment

$$(X,Y) \to W(X,Y)$$

satisfies all the requirements imposed on a covariant derivative $\nabla_X Y$. □

Consider (3.4.27) with $X = \partial_\gamma$, $Y = \partial_\beta$ and $Z = \partial_\sigma$,

$$\begin{aligned} 2g(\nabla_\gamma \partial_\beta, \partial_\sigma) &= 2g(\Gamma^\rho{}_{\beta\gamma}\partial_\rho, \partial_\sigma) \\ &= 2g_{\rho\sigma}\Gamma^\rho{}_{\beta\gamma} \\ &= \partial_\gamma g_{\beta\sigma} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma} \end{aligned}$$

Multiplying this equation by $g^{\alpha\sigma}/2$ we then obtain

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}\{\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\}} . \tag{3.4.29}$$

3.4.4 Geodesics and Christoffel symbols

A twice-differentiable curve $\gamma[a,b] \to M$ is said to be a *geodesic* if it solves the equation[2]

$$\nabla_{\dot\gamma}\dot\gamma = 0 . \tag{3.4.30}$$

One says that "$\dot\gamma$ is parallelly propaged along γ". An alternative, equivalent approach, is to require γ to be a stationary point of the action

$$I(\gamma) = \int_a^b \underbrace{\frac{1}{2} g(\dot\gamma,\dot\gamma)(s)}_{=:\mathscr{L}(\gamma,\dot\gamma)} ds . \tag{3.4.31}$$

Thus,

$$\mathscr{L}(x^\mu, \dot x^\nu) = \frac{1}{2} g_{\alpha\beta}(x^\mu)\dot x^\alpha \dot x^\beta .$$

One readily finds the Euler-Lagrange equations for $\mathscr{L}$:

$$\frac{d}{ds}\left(\frac{\partial\mathscr{L}}{\partial \dot x^\mu}\right) = \frac{\partial \mathscr{L}}{\partial x^\mu} \quad \Longleftrightarrow \quad \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 . \tag{3.4.32}$$

[2]Strictly speaking, this should be called a *geodesic segment*, the name "geodesic" being reserved to maximally extended solutions of this (3.4.30); however, we shall not make the distinction between geodesics and geodesic segments unless it is essential to do so.

This is clearly identical to (3.4.30).

It turns out that the left member of the equivalence in (3.4.32) provides a very convenient way of calculating the Christoffel symbols: given a metric g, write down $\mathscr{L}$, work out the Euler-Lagrange equations, and identify the Christoffels as the coefficients of the first derivative terms in those equations; see Example 3.5.2 below for an application.

The Euler-Lagrange equations for (3.4.31) are identical with those of

$$\tilde{I}(\gamma) = \int_a^b \sqrt{|g(\dot\gamma,\dot\gamma)(s)|}ds \;, \qquad (3.4.33)$$

but (3.4.31) is more convenient to work with. Note also that $\mathscr{L}$ is differentiable at points where $\dot\gamma$ vanishes, while $\sqrt{|g(\dot\gamma,\dot\gamma)(s)|}$ is not. The aesthetic advantage of (3.4.33), of being reparameterization-invariant, is more than compensated by the calculational convenience of $\mathscr{L}$.

3.5 Curvature

Let ∇ be a covariant derivative defined for vector fields, the curvature tensor is defined by the formula

$$\boxed{R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z}\;, \qquad (3.5.1)$$

where, as elsewhere, $[X,Y]$ is the Lie bracket defined in (3.1.4). We note the anti-symmetry

$$R(X,Y)Z = -R(Y,X)Z\;. \qquad (3.5.2)$$

It turns out the this defines a tensor. Multi-linearity with respect to addition is obvious, but multiplication by functions require more work.

First, we have (see (3.4.21))

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z \\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \underbrace{\nabla_{f[X,Y]-Y(f)X}Z}_{=f\nabla_{[X,Y]}Z-Y(f)\nabla_X Z} \\ &= fR(X,Y)Z\;. \end{aligned}$$

The simplest proof of linearity in the last slot proceeds via an index calculation in adapted coordinates; so while we will do the "elegant", index-free version shortly, let us do the ugly one first. We use the coordinate system of Proposition 3.5.3 below, in which the first derivatives of the metric vanish at the prescribed point p:

$$\begin{aligned} \nabla_i\nabla_j Z^k &= \partial_i(\partial_j Z^k - \Gamma^k{}_{\ell j}Z^\ell) + \underbrace{0\times\nabla Z}_{\text{at } p} \\ &= \partial_i\partial_j Z^k - \partial_i\Gamma^k{}_{\ell j}Z^\ell \quad \text{at } p\;. \end{aligned} \qquad (3.5.3)$$

Antisymmetrising in i and j, the terms involving the second derivatives of Z drop out, so the result is indeed linear in Z. So $\nabla_i\nabla_j Z^k - \nabla_j\nabla_i Z^k$ is a tensor field linear in Z, and therefore can be written as $R^k{}_{\ell ij}Z^\ell$.

Note that $\nabla_i\nabla_j Z^k$ is, by definition, the tensor field of first covariant derivatives of the tensor field $\nabla_j Z^k$, and it isn't completely obvious that this is the same as what occurs in (3.5.1), so this argument requires a further justification.

Next,

$$\begin{aligned}
R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ)\\
&= \Big\{\nabla_X\Big(Y(f)Z + f\nabla_Y Z\Big)\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y}\\
&\quad -[X,Y](f)Z - f\nabla_{[X,Y]}Z\\
&= \Big\{\underbrace{X(Y(f))Z}_{a} + \underbrace{Y(f)\nabla_X Z + X(f)\nabla_Y Z}_{b} + f\nabla_X\nabla_Y Z\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y}\\
&\quad - \underbrace{[X,Y](f)Z}_{c} - f\nabla_{[X,Y]}Z\ .
\end{aligned}$$

Now, a together with its counterpart with X and Y interchanged cancel out with c, while b is symmetric with respect to X and Y and therefore cancels out with its counterpart with X and Y interchanged, leading to the desired equality

$$R(X,Y)(fZ) \quad = \quad fR(X,Y)Z\ .$$

In a coordinate basis $\{e_a\} = \{\partial_\mu\}$ we find[3] (recall that $[\partial_\mu, \partial_\nu] = 0$)

$$\begin{aligned}
R^\alpha{}_{\beta\gamma\delta} &:= \langle dx^\alpha, R(\partial_\gamma, \partial_\delta)\partial_\beta\rangle\\
&= \langle dx^\alpha, \nabla_\gamma\nabla_\delta\partial_\beta\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma}\\
&= \langle dx^\alpha, \nabla_\gamma(\Gamma^\sigma{}_{\beta\delta}\partial_\sigma)\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma}\\
&= \langle dx^\alpha, \partial_\gamma(\Gamma^\sigma{}_{\beta\delta})\partial_\sigma + \Gamma^\rho{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\partial_\rho\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma}\\
&= \{\partial_\gamma\Gamma^\alpha{}_{\beta\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\} - \{\cdots\}_{\delta\leftrightarrow\gamma}\ ,
\end{aligned}$$

leading finally to

$$\boxed{\boxed{R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}}}\ . \tag{3.5.4}$$

In a general frame some supplementary commutator terms will appear in the formula for $R^a{}_{bcd}$.

We note the following:

THEOREM 3.5.1 *There exists a coordinate system in which the metric tensor field has vanishing second derivatives at p if and only if its Riemann tensor vanishes at p. Furthermore, there exists a coordinate system in which the metric tensor field has constant entries near p if and only if the Riemann tensor vanishes near p.*

PROOF: The condition is necessary, since Riem is a tensor. The sufficiency will be admitted. □

The calculation of the curvature tensor is often a very traumatic experience. There is one obvious case where things are painless, when all $g_{\mu\nu}$'s are constants: in this case the Christoffels vanish, and so does the curvature tensor.

For more general metrics one way out is to use symbolic computer algebra, e.g. on `http://grtensor.phy.queensu.ca/NewDemo`.

[3]The reader is warned that certain authors use a different sign convention either for $R(X,Y)Z$, or for $R^\alpha{}_{\beta\gamma\delta}$, or both. A useful table that lists the sign conventions for a series of standard GR references can be found on the backside of the front cover of [48].

EXAMPLE 3.5.2 As a less trivial example, consider the round two sphere, which we write in the form

$$g = d\theta^2 + e^{2f} d\varphi^2 \;, \qquad e^{2f} = \sin^2\theta \;.$$

The Christoffel symbols are easily founds from the Lagrangean for geodesics:

$$\mathscr{L} = \frac{1}{2}(\dot\theta^2 + e^{2f}\dot\varphi^2) \;.$$

The Euler-Lagrange equations give

$$\Gamma^\theta{}_{\varphi\varphi} = -f'e^{2f} \;, \quad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = f' \;,$$

with the remaining Christoffel symbols vanishing. Using the definition of the Riemann tensor we then immediately find

$$R^\varphi{}_{\theta\varphi\theta} = -f'' - (f')^2 = 1 \;.$$

All remaining components of the Riemann tensor can be obtained from this one by raising and lowering of indices, together with the symmetry operations which we are about to describe. This leads to

$$R_{ab} = g_{ab} \;, \qquad R = 2 \;.$$

Equation (3.5.1) is most frequently used "upside-down", not as a definition of the Riemann tensor, but as a tool for calculating what happens when one changes the order of covariant derivatives. Recall that for partial derivatives we have

$$\partial_\mu\partial_\nu Z^\sigma = \partial_\nu\partial_\mu Z^\sigma \;,$$

but this is not true in general if partial derivatives are replaced by covariant ones:

$$\nabla_\mu\nabla_\nu Z^\sigma \neq \nabla_\nu\nabla_\mu Z^\sigma \;.$$

To find the correct formula let us consider the tensor field S defined as

$$Y \longrightarrow S(Y) := \nabla_Y Z \;.$$

In local coordinates, S takes the form

$$S = \nabla_\mu Z^\nu \, dx^\mu \otimes \partial_\nu \;.$$

It follows from the Leibniz rule — or, equivalently, from the definitions in Section 3.4 — that we have

$$\begin{aligned}(\nabla_X S)(Y) &= \nabla_X(S(Y)) - S(\nabla_X Y)\\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y} Z \;.\end{aligned}$$

The commutator of the derivatives can then be calculated as

$$\begin{aligned}(\nabla_X S)(Y) - (\nabla_Y S)(X) &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z\\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z\\ &\quad + \nabla_{[X,Y]} Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z\\ &= R(X,Y)Z - \nabla_{T(X,Y)} Z \;. \qquad (3.5.5)\end{aligned}$$

Writing ∇S in the usual form

$$\nabla S = \nabla_\sigma S_\mu{}^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu = \nabla_\sigma \nabla_\mu Z^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu \ ,$$

we are thus led to

$$\nabla_\mu \nabla_\nu Z^\alpha - \nabla_\nu \nabla_\mu Z^\alpha = R^\alpha{}_{\sigma\mu\nu} Z^\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma Z^\alpha \ . \tag{3.5.6}$$

In the important case of vanishing torsion, the coordinate-component equivalent of (3.5.1) is thus

$$\boxed{\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\sigma\mu\nu} X^\sigma} \ . \tag{3.5.7}$$

An identical calculation gives, still for torsionless connections,

$$\nabla_\mu \nabla_\nu a_\alpha - \nabla_\nu \nabla_\mu a_\alpha = -R^\sigma{}_{\alpha\mu\nu} a_\sigma \ . \tag{3.5.8}$$

For a general tensor t and torsion-free connection each tensor index comes with a corresponding Riemann tensor term:

$$\begin{aligned}
&\nabla_\mu \nabla_\nu t_{\alpha_1\ldots\alpha_r}{}^{\beta_1\ldots\beta_s} - \nabla_\nu \nabla_\mu t_{\alpha_1\ldots\alpha_r}{}^{\beta_1\ldots\beta_s} = \\
&\quad -R^\sigma{}_{\alpha_1\mu\nu} t_{\sigma\ldots\alpha_r}{}^{\beta_1\ldots\beta_s} - \ldots - R^\sigma{}_{\alpha_r\mu\nu} t_{\alpha_1\ldots\sigma}{}^{\beta_1\ldots\beta_s} \\
&\quad +R^{\beta_1}{}_{\sigma\mu\nu} t_{\alpha_1\ldots\alpha_r}{}^{\sigma\ldots\beta_s} + \ldots + R^{\beta_s}{}_{\sigma\mu\nu} t_{\alpha_1\ldots\alpha_r}{}^{\beta_1\ldots\sigma} \ .
\end{aligned} \tag{3.5.9}$$

3.5.1 Bianchi identities

We have already seen the anti-symmetry property of the Riemann tensor, which in the index notation corresponds to the equation

$$R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma} \ . \tag{3.5.10}$$

There are a few other identities satisfied by the Riemann tensor, we start with the *first Bianchi identity*. Let $A(X,Y,Z)$ be any expression depending upon three vector fields X,Y,Z which is antisymmetric in X and Y, we set

$$\sum_{[XYZ]} A(X,Y,Z) := A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y) \ , \tag{3.5.11}$$

thus $\sum_{[XYZ]}$ is a sum over cyclic permutations of the vectors X,Y,Z. Clearly,

$$\sum_{[XYZ]} A(X,Y,Z) = \sum_{[XYZ]} A(Y,Z,X) = \sum_{[XYZ]} A(Z,X,Y) \ . \tag{3.5.12}$$

Suppose, first, that X, Y and Z commute. Using (3.5.12) together with the definition (3.4.18) of the torsion tensor T we calculate

$$\begin{aligned}
\sum_{[XYZ]} R(X,Y)Z &= \sum_{[XYZ]} \Big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \Big) \\
&= \sum_{[XYZ]} \Big(\nabla_X \nabla_Y Z - \nabla_Y \underbrace{(\nabla_Z X + T(X,Z))}_{\text{we have used } [X,Z]=0, \text{ see (3.4.18)}} \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\sum_{[XYZ]} \nabla_X \nabla_Y Z - \sum_{[XYZ]} \nabla_Y \nabla_Z X}_{=0 \text{ (see (3.5.12))}} - \sum_{[XYZ]} \nabla_Y (\underbrace{T(X,Z)}_{=-T(Z,X)}) \\
&= \sum_{[XYZ]} \nabla_X (T(Y,Z)) \, ,
\end{aligned}
$$

and in the last step we have again used (3.5.12). This can be somewhat rearranged by using the definition of the covariant derivative of a higher order tensor (compare (3.4.16)) — equivalently, using the Leibniz rule rewritten upside-down:

$$
(\nabla_X T)(Y,Z) = \nabla_X (T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z) \, .
$$

This leads to

$$
\begin{aligned}
\sum_{[XYZ]} \nabla_X (T(Y,Z)) &= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(\nabla_X Y, Z) + T(Y, \underbrace{\nabla_X Z}_{=T(X,Z)+\nabla_Z X}) \Big) \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) - T(\underbrace{T(X,Z)}_{=-T(Z,X)}, Y) \Big) \\
&\quad \underbrace{+ \sum_{[XYZ]} T(\nabla_X Y, Z) + \sum_{[XYZ]} \underbrace{T(Y, \nabla_Z X)}_{=-T(\nabla_Z X, Y)}}_{=0 \text{ (see (3.5.12))}} \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y), Z) \Big) \, .
\end{aligned}
$$

Summarizing, we have obtained the first Bianchi identity:

$$
\sum_{[XYZ]} R(X,Y)Z = \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z) \Big) \, , \tag{3.5.13}
$$

under the hypothesis that X, Y and Z commute. However, both sides of this equation are tensorial with respect to X, Y and Z, so that they remain correct without the commutation hypothesis.

We are mostly interested in connections with vanishing torsion, in which case (3.5.13) can be rewritten as

$$
\boxed{R^\alpha{}_{\beta\gamma\delta} + R^\alpha{}_{\gamma\delta\beta} + R^\alpha{}_{\delta\beta\gamma} = 0} \, . \tag{3.5.14}
$$

Our next goal is the *second Bianchi identity*. We consider four vector fields X, Y, Z and W and we assume again that everybody commutes with everybody else. We calculate

$$
\begin{aligned}
\sum_{[XYZ]} \nabla_X (R(Y,Z)W) &= \sum_{[XYZ]} \Big(\underbrace{\nabla_X \nabla_Y \nabla_Z W}_{=R(X,Y)\nabla_Z W + \nabla_Y \nabla_X \nabla_Z W} - \nabla_X \nabla_Z \nabla_Y W \Big) \\
&= \sum_{[XYZ]} R(X,Y) \nabla_Z W
\end{aligned}
$$

$$+\underbrace{\sum_{[XYZ]}\nabla_Y\nabla_X\nabla_Z W-\sum_{[XYZ]}\nabla_X\nabla_Z\nabla_Y W}_{=0}$$

$$. \tag{3.5.15}$$

Next,

$$\begin{aligned}\sum_{[XYZ]}(\nabla_X R)(Y,Z)W &= \sum_{[XYZ]}\Big(\nabla_X(R(Y,Z)W)-R(\nabla_X Y,Z)W \\ &\quad -R(Y,\underbrace{\nabla_X Z}_{=\nabla_Z X+T(X,Z)})W-R(Y,Z)\nabla_X W\Big) \\ &= \sum_{[XYZ]}\nabla_X(R(Y,Z)W) \\ &\quad \underbrace{-\sum_{[XYZ]}R(\nabla_X Y,Z)W-\sum_{[XYZ]}\underbrace{R(Y,\nabla_Z X)W}_{=-R(\nabla_Z X,Y)W}}_{=0} \\ &\quad -\sum_{[XYZ]}\Big(R(Y,T(X,Z))W+R(Y,Z)\nabla_X W\Big) \\ &= \sum_{[XYZ]}\Big(\nabla_X(R(Y,Z)W)-R(T(X,Y),Z)W-R(Y,Z)\nabla_X W\Big)\,.\end{aligned}$$

It follows now from (3.5.15) that the first term cancels out the third one, leading to

$$\sum_{[XYZ]}(\nabla_X R)(Y,Z)W=-\sum_{[XYZ]}R(T(X,Y),Z)W\,, \tag{3.5.16}$$

which is the desired second Bianchi identity for commuting vector fields. As before, because both sides are multi-linear with respect to addition and multiplication by functions, the result remains valid for arbitrary vector fields.

For torsionless connections the components equivalent of (3.5.16) reads

$$\boxed{R^\alpha{}_{\mu\beta\gamma;\delta}+R^\alpha{}_{\mu\gamma\delta;\beta}+R^\alpha{}_{\mu\delta\beta;\gamma}=0}\,. \tag{3.5.17}$$

3.5.2 Pair interchange symmetry

There is one more identity satisfied by the curvature tensor which is specific to the curvature tensor associated with the Levi-Civita connection, namely

$$g(X,R(Y,Z)W)=g(Y,R(X,W)Z)\,. \tag{3.5.18}$$

If one sets

$$\boxed{R_{abcd}:=g_{ae}R^e{}_{bcd}}\,, \tag{3.5.19}$$

then (3.5.18) is equivalent to

$$\boxed{R_{abcd}=R_{cdab}}\,. \tag{3.5.20}$$

We will present two proofs of (3.5.18). The first is direct, but not very elegant. The second is prettier, but less insightful.

For the ugly proof, it is convenient to first establish some preliminary results, which are of interest on their own:

PROPOSITION 3.5.3 *1. Let g be a continuous Lorentzian metric, for every $p \in M$ there exists a neighborhood thereof with a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,+1,\cdots,+1)$ at p.*

2. If g is differentiable, then the coordinates can be further chosen so that

$$\partial_\sigma g_{\alpha\beta} = 0 \tag{3.5.21}$$

at p, while preserving the degree of differentiability of g.

REMARK 3.5.4 *The properties spelled-out above do of course hold in the normal coordinates. However, the introduction of normal coordinates does lead to a loss of differentiability of the metric.*

PROOF: 1. Let y^μ be any coordinate system around p, shifting by a constant vector we can assume that p corresponds to $y^\mu = 0$. Let $e_a = e_a{}^\mu \partial/\partial y^\mu$ be any frame at p such that $g(e_a, e_b) = \eta_{ab}$ — such frames can be found by, *e.g.*, a Gram-Schmidt orthogonalisation. Calculating the determinant of both sides of the equation

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}$$

we obtain, at p,

$$\det(g_{\mu\nu}) \det(e_a{}^\mu)^2 = -1 \ ,$$

which shows that $\det(e_a{}^\mu)$ is non-vanishing. It follows that the formula

$$y^\mu = e^\mu{}_a x^a$$

defines a (linear) diffeomorphism. In the new coordinates we have, again at p,

$$g\Big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\Big) = e^\mu{}_a e^\nu{}_b g\Big(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\Big) = \eta_{ab} \ . \tag{3.5.22}$$

2. Let x^μ be the coordinates described in point 1., note that p lies at the origin of those coordinates. The new coordinates z^α will be implicitly defined by the equations

$$x^\mu = z^\mu + \frac{1}{2} A^\mu{}_{\alpha\beta} z^\alpha z^\beta \ ,$$

where $A^\mu{}_{\alpha\beta}$ is a set of constants, symmetric with respect to the interchange of α and β . Set

$$g'_{\alpha\beta} := g\Big(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\Big) \ , \qquad g_{\alpha\beta} := g\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\Big) \ .$$

Recall the transformation law

$$g'_{\mu\nu}(z^\sigma) = g_{\alpha\beta}(x^\rho(z^\sigma)) \frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial x^\beta}{\partial z^\nu} \ .$$

By differentiation one obtains at $x^\mu = z^\mu = 0$,

$$\begin{aligned}\frac{\partial g'_{\mu\nu}}{\partial z^\rho}(0) &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + g_{\alpha\beta}(0)\Big(A^\alpha{}_{\mu\rho}\delta^\beta_\nu + \delta^\alpha_\mu A^\beta{}_{\nu\rho}\Big)\\ &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + A_{\nu\mu\rho} + A_{\mu\nu\rho}\ ,\end{aligned} \tag{3.5.23}$$

where

$$A_{\alpha\beta\gamma} = g_{\alpha\sigma}(0)A^\sigma{}_{\beta\gamma}\ .$$

It remains to show that we can choose $A^\sigma{}_{\beta\gamma}$ so that the left-hand-side can be made to vanish at p. An explicit formula for $A_{\sigma\beta\gamma}$ can be obtained from (3.5.23) by a cyclic permutation calculation similar to that in (3.4.26). After raising the first index, the final result is

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}\left\{\frac{\partial g_{\beta\gamma}}{\partial x^\rho} - \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial x^\beta}\right\}(0)\ ;$$

the reader may wish to check directly that this does indeed lead to a vanishing right-hand-side of (3.5.23). □

We are ready now to pass to the proof of (3.5.20). We suppose that the metric is twice-differentiable, by point 2. of Proposition 3.5.3 in a neighborhood of any point $p \in M$ there exists a coordinate system in which the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ vanish at p. Equation (3.5.4) evaluated at p therefore reads

$$\begin{aligned}R^\alpha{}_{\beta\gamma\delta} &= \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma}\\ &= \frac{1}{2}\Big\{g^{\alpha\sigma}\partial_\gamma(\partial_\delta g_{\sigma\beta} + \partial_\beta g_{\sigma\delta} - \partial_\sigma g_{\beta\delta})\\ &\quad -g^{\alpha\sigma}\partial_\delta(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma})\Big\}\\ &= \frac{1}{2}g^{\alpha\sigma}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}\ .\end{aligned}$$

Equivalently,

$$R_{\sigma\beta\gamma\delta}(0) = \frac{1}{2}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}(0)\ . \tag{3.5.24}$$

This last expression is obviously symmetric under the exchange of $\sigma\beta$ with $\gamma\delta$, leading to (3.5.20).

The above calculation traces back the pair-interchange symmetry to the definition of the Levi-Civita connection in terms of the metric tensor. As already mentioned, there exists a more elegant proof, where the origin of the symmetry is perhaps somewhat less apparent, which proceeds as follows: We start by noting that

$$0 = \nabla_a\nabla_b g_{cd} - \nabla_b\nabla_a g_{cd} = -R^e{}_{cab}g_{ed} - R^e{}_{dab}g_{ce}\ , \tag{3.5.25}$$

leading to anti-symmetry in the first two indices:

$$R_{abcd} = -R_{bacd}\ .$$

Next, using the cyclic symmetry for a torsion-free connection, we have

$$\begin{aligned} R_{abcd} + R_{cabd} + R_{bcad} &= 0 \,, \\ R_{bcda} + R_{dbca} + R_{cdba} &= 0 \,, \\ R_{cdab} + R_{acdb} + R_{dacb} &= 0 \,, \\ R_{dabc} + R_{bdac} + R_{abdc} &= 0 \,. \end{aligned}$$

The desired equation (3.5.20) follows now by adding the first two and subtracting the last two equations, using (3.5.25).

It is natural to enquire about the number of independent components of a tensor with the symmetries of a metric Riemann tensor in dimension n, the calculation proceeds as follows: as R_{abcd} is symmetric under the exchange of ab with cd, and anti-symmetric in each of these pairs, we can view it as a symmetric map from the space of anti-symmetric tensor with two indices. Now, the space of anti-symmetric tensors is $N = n(n-1)/2$ dimensional, while the space of symmetric maps in dimension N is $N(N+1)/2$ dimensional, so we obtain at most $n(n-1)(n^2-n+2)/8$ free parameters. However, we need to take into account the cyclic identity:

$$R_{abcd} + R_{bcad} + R_{cabd} = 0 \,. \tag{3.5.26}$$

If $a = b$ this reads

$$R_{aacd} + R_{acad} + R_{caad} = 0 \,,$$

which has already been accounted for. Similarly if $a = d$ we obtain

$$R_{abca} + R_{bcaa} + R_{caba} = 0 \,,$$

which holds in view of the previous identities. We conclude that the only new identities which could possibly arise are those where $abcd$ are all distinct. Clearly no expression involving three such components of the Riemann tensor can be obtained using the previous identities, so this is an independent constraint. In dimension four (3.5.26) provides thus four candidate equations for another constraint, labeled by d, but it is easily checked that they all coincide; this leads to 20 free parameters at each space point. The reader is encouraged to finish the counting in higher dimensions.

3.6 Geodesic deviation (Jacobi equation)

Suppose that we have a one parameter family of geodesics $\gamma(s, \lambda)$, where s is the parameter along the geodesic, and λ is a parameter which distinguishes the geodesics. Set

$$Z(s,\lambda) := \frac{\partial \gamma(s,\lambda)}{\partial \lambda} \equiv \frac{\partial \gamma^i(s,\lambda)}{\partial \lambda}\partial_i \,,$$

for each λ this defines a vector field Z along $\gamma(\lambda)$, which measures how nearby geodesics deviate from each other, since, to first order,

$$\gamma^i(s,\lambda) = \gamma^i(s,\lambda_0) + Z^i(\lambda - \lambda_0) + O((\lambda-\lambda_0)^2) \,.$$

To measure how a vector field W changes along $s \to \gamma(s, \lambda)$ one introduces the operator

$$\frac{DW^\mu}{ds} \quad := \quad \frac{\partial W^\mu}{\partial s} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha \tag{3.6.1}$$

$$= \dot\gamma^\beta \frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha \tag{3.6.2}$$
$$= \dot\gamma^\beta \nabla_\beta W^\mu \,. \tag{3.6.3}$$

(It would perhaps be more logical to write $\frac{DW^\mu}{\partial s}$, but we will stick to the previous notation.) Analogously we define

$$\frac{DW^\mu}{d\lambda} := \frac{\partial W^\mu}{\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha \tag{3.6.4}$$
$$= \partial_\lambda\gamma^\beta \frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha \tag{3.6.5}$$
$$= Z^\beta \nabla_\beta W^\mu \,. \tag{3.6.6}$$

Note that since $s \to \gamma(s,\lambda)$ is a geodesic we have from (3.6.1) and (3.6.3)

$$\frac{D^2\gamma^\mu}{ds^2} := \frac{D\dot\gamma^\mu}{ds} = \dot\gamma^\alpha\nabla_\alpha\dot\gamma^\mu = \frac{\partial^2\gamma^\mu}{\partial s} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\dot\gamma^\alpha = 0 \,. \tag{3.6.7}$$

Further,

$$\nabla_{\dot\gamma} Z^\mu = \dot\gamma^\nu\nabla_\nu Z^\mu = \dot\gamma^\nu\nabla_\nu\partial_\lambda\gamma^\mu \underbrace{=}_{(3.6.3)} \frac{\partial^2\gamma^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\partial_\lambda\gamma^\alpha \underbrace{=}_{(3.6.6)} Z^\beta\nabla_\beta\dot\gamma^\mu = \nabla_Z\dot\gamma^\mu \,. \tag{3.6.8}$$

In other words,

$$\nabla_{\dot\gamma} Z = \nabla_Z \dot\gamma \,. \tag{3.6.9}$$

One then calculates as follows:

$$\begin{aligned}
\frac{D^2Z^\mu}{ds^2}(s) &= \dot\gamma^\alpha\nabla_\alpha(\dot\gamma^\beta\nabla_\beta Z^\mu) \\
&= \dot\gamma^\alpha\nabla_\alpha(Z^\beta\nabla_\beta\dot\gamma^\mu) \\
&= (\dot\gamma^\alpha\nabla_\alpha Z^\beta)\nabla_\beta\dot\gamma^\mu + Z^\beta\dot\gamma^\alpha\nabla_\alpha\nabla_\beta\dot\gamma^\mu \\
&= (\dot\gamma^\alpha\nabla_\alpha Z^\beta)\nabla_\beta\dot\gamma^\mu + Z^\beta\dot\gamma^\alpha(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\dot\gamma^\mu + Z^\beta\dot\gamma^\alpha\nabla_\beta\nabla_\alpha\dot\gamma^\mu \\
&= (\dot\gamma^\alpha\nabla_\alpha Z^\beta)\nabla_\beta\dot\gamma^\mu + Z^\beta\dot\gamma^\alpha R_{\alpha\beta\sigma}{}^\mu\dot\gamma^\sigma + Z^\beta\dot\gamma^\alpha\nabla_\beta\nabla_\alpha\dot\gamma^\mu \\
&= (\dot\gamma^\alpha\nabla_\alpha Z^\beta)\nabla_\beta\dot\gamma^\mu + Z^\beta\dot\gamma^\alpha R_{\alpha\beta\sigma}{}^\mu\dot\gamma^\sigma + Z^\beta\nabla_\beta(\underbrace{\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\mu}_{0}) - (Z^\beta\nabla_\beta\dot\gamma^\alpha)\nabla_\alpha\dot\gamma^\mu \,.
\end{aligned}$$

The first and the last term cancel out by (3.6.9), resulting in

$$\frac{D^2Z^\mu}{ds^2}(s) = R_{\alpha\beta\sigma}{}^\mu\dot\gamma^\alpha Z^\beta\dot\gamma^\sigma \,. \tag{3.6.10}$$

We have obtained an equation known as the *Jacobi equation*, or as the *geodesic deviation equation*:

$$\boxed{\frac{D^2Z}{ds^2} = R(\dot\gamma, Z)\dot\gamma} \,. \tag{3.6.11}$$

Solutions of (3.6.11) are called *Jacobi fields* along γ.

3.7 Moving frames

A formalism which is very convenient for practical calculations is that of *moving frames*; it also plays a key role when considering spinors. By definition, a moving frame is a (locally defined) field of bases $\{e_a\}$ of TM such that the scalar products

$$g_{ab} := g(e_a, e_b) \tag{3.7.1}$$

are point independent. In most standard applications one assumes that the e_a's form an orthonormal basis, so that g_{ab} is a diagonal matrix with plus and minus ones on the diagonal. However, it is sometimes convenient to allow other such frames, *e.g.* with isotropic vectors being members of the frame.

It is customary to denote by ${\omega^a}_{bc}$ the associated connection coefficients:

$${\omega^a}_{bc} := \theta^a(\nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = {\omega^a}_{bc} X^c e_a \ , \tag{3.7.2}$$

where, as elsewhere, $\{\theta^a(p)\}$ is a basis of T^*_pM dual to $\{e_a(p)\} \subset T_pM$; we will refer to θ^a as a *coframe*. The *connection one forms* ${\omega^a}_b$ are defined as

$${\omega^a}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = {\omega^a}_b(X) e_a \ . \tag{3.7.3}$$

As always we use the metric to raise and lower indices, so that

$$\omega_{abc} := g_{ad}{\omega^e}_{bc} \ , \quad \omega_{ab} := g_{ae}{\omega^e}_b \ . \tag{3.7.4}$$

When ∇ is metric compatible, the ω_{ab}'s are anti-antisymmetric: indeed, as the g_{ab}'s are point independent, for any vector field X we have

$$\begin{aligned}
0 = X(g_{ab}) = X(g(e_a, e_b)) &= g(\nabla_X e_a, e_b) + g(e_a, \nabla_X e_b) \\
&= g({\omega^c}_a(X) e_c, e_b) + g(e_a, {\omega^d}_b(X) e_d) \\
&= g_{cb}{\omega^c}_a(X) + g_{ad}{\omega^d}_b(X) \\
&= \omega_{ba}(X) + \omega_{ab}(X) \ .
\end{aligned}$$

Hence

$$\boxed{\omega_{ab} = -\omega_{ba} \quad \Longleftrightarrow \quad \omega_{abc} = -\omega_{bac}} \ . \tag{3.7.5}$$

If the connection is the Levi-Civita connection of g, this equation will allow us to algebraically express the ω_{ab}'s in terms of the Lie brackets of the vector fields e_a. In order to see this, we note that

$$g(e_a, \nabla_{e_c} e_b) = g(e_a, {\omega^d}_{bc} e_d) = g_{ad}{\omega^d}_{bc} = \omega_{abc} \ .$$

Rewritten the other way round this gives an alternative equation for the ω's with all indices down:

$$\omega_{abc} = g(e_a, \nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \omega_{ab}(X) = g(e_a, \nabla_X e_b) \ . \tag{3.7.6}$$

If ∇ has no torsion we find

$$\omega_{abc} - \omega_{acb} = g(e_a, \nabla_{e_c} e_b - \nabla_{e_b} e_c) = g(e_a, [e_c, e_b]) \ .$$

We can now do the usual cyclic permutation calculation to obtain

$$\begin{aligned} \omega_{abc} - \omega_{acb} &= g(e_a, [e_c, e_b]) \ , \\ -(\omega_{bca} - \omega_{bac}) &= -g(e_b, [e_a, e_c]) \ , \\ -(\omega_{cab} - \omega_{cba}) &= -g(e_c, [e_b, e_a]) \ . \end{aligned}$$

Summing the three equations and using (3.7.5) we obtain

$$\boxed{\omega_{abc} = \frac{1}{2}\Big(g(e_a, [e_c, e_b]) - g(e_b, [e_a, e_c]) - g(e_c, [e_b, e_a])\Big)} \ . \tag{3.7.7}$$

Equation (3.7.7) provides an explicit expression for the ω's. While it is useful to know that there is one, and while this expression is useful to estimate things, it is rarely used for practical calculations; see Example 3.7.1 for more comments about that last issue.

It turns out that one can obtain a simple expression for the torsion of ω using exterior differentiation. Recall that if α is a one-form, then its exterior derivative $d\alpha$ can be defined using the formula

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \ . \tag{3.7.8}$$

We set

$$T^a(X, Y) := \theta^a(T(X, Y)) \ ,$$

and using (3.7.8) together with the definition (3.4.18) of the torsion tensor T we calculate as follows:

$$\begin{aligned} T^a(X, Y) &= \theta^a(\nabla_X Y - \nabla_Y X - [X, Y]) \\ &= X(Y^a) + \omega^a{}_b(X)Y^b - Y(X^a) - \omega^a{}_b(Y)X^b - \theta^a([X, Y]) \\ &= X(\theta^a(Y)) - Y(\theta^a(X)) - \theta^a([X, Y]) + \omega^a{}_b(X)\theta^b(Y) - \omega^a{}_b(Y)\theta^b(X) \\ &= d\theta^a(X, Y) + (\omega^a{}_b \wedge \theta^b)(X, Y) \ . \end{aligned}$$

It follows that

$$T^a = d\theta^a + \omega^a{}_b \wedge \theta^b \ . \tag{3.7.9}$$

In particular when the torsion vanishes we obtain the so-called *Cartan's first structure equation*

$$\boxed{d\theta^a + \omega^a{}_b \wedge \theta^b = 0} \ . \tag{3.7.10}$$

EXAMPLE 3.7.1 As an example of the moving frame technique we consider (the most general) three-dimensional spherically symmetric metric

$$g = e^{2\beta(r)}dr^2 + e^{2\gamma(r)}d\theta^2 + e^{2\gamma(r)}\sin^2\theta d\varphi^2 \ . \tag{3.7.11}$$

There is an obvious choice of ON coframe for g given by

$$\theta^1 = e^{\beta(r)}dr \ , \ \theta^2 = e^{\gamma(r)}d\theta \ , \ \theta^3 = e^{\gamma(r)}\sin\theta d\varphi \ , \tag{3.7.12}$$

leading to

$$g = \theta^1 \otimes \ \theta^1 + \theta^2 \otimes \ \theta^2 + \theta^3 \otimes \ \theta^3 \ ,$$

so that the frame e_a dual to the θ^a's will be ON, as desired:

$$g_{ab} = g(e_a, e_b) = \mathrm{diag}(1,1,1) \ .$$

The idea of the calculation which we are about to do is the following: there is only one connection which is compatible with the metric, and which is torsion free. If we find a set of one forms ω_{ab} which exhibit the properties just mentioned, then they have to be the connection forms of the Levi-Civita connection. As shown in the calculation leading to (3.7.5), the compatibility with the metric will be ensured if we require

$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \ ,$$

$$\omega_{12} = -\omega_{21} \ , \quad \omega_{13} = -\omega_{31} \ , \quad \omega_{23} = -\omega_{32} \ .$$

Next, we have the equations for the vanishing of torsion:

$$\begin{aligned}
0 = d\theta^1 &= -\underbrace{\omega^1{}_1}_{=0}\theta^1 - \omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \\
&= -\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \ , \\
d\theta^2 &= \gamma' e^\gamma dr \wedge d\theta = \gamma' e^{-\beta}\theta^1 \wedge \theta^2 \\
&= -\underbrace{\omega^2{}_1}_{=-\omega^1{}_2}\theta^1 - \underbrace{\omega^2{}_2}_{=0}\theta^2 - \omega^2{}_3\theta^3 \\
&= \omega^1{}_2\theta^1 - \omega^2{}_3\theta^3 \ , \\
d\theta^3 &= \gamma' e^\gamma \sin\theta\, dr \wedge d\varphi + e^\gamma \cos\theta\, d\theta \wedge d\varphi = \gamma' e^{-\beta}\theta^1 \wedge \theta^3 + e^{-\gamma}\cot\theta\, \theta^2 \wedge \theta^3 \\
&= -\underbrace{\omega^3{}_1}_{=-\omega^1{}_3}\theta^1 - \underbrace{\omega^3{}_2}_{=-\omega^2{}_3}\theta^2 - \underbrace{\omega^3{}_3}_{=0}\theta^3 \\
&= \omega^1{}_3\theta^1 + \omega^2{}_3\theta^2 \ .
\end{aligned}$$

Summarising,

$$\begin{aligned}
-\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \qquad\qquad &= 0 \ , \\
\omega^1{}_2\theta^1 \qquad - \omega^2{}_3\theta^3 &= \gamma' e^{-\beta}\theta^1 \wedge \theta^2 \ , \\
\omega^1{}_3\theta^1 + \omega^2{}_3\theta^2 &= \gamma' e^{-\beta}\theta^1 \wedge \theta^3 + e^{-\gamma}\cot\theta\, \theta^2 \wedge \theta^3 \ .
\end{aligned}$$

It should be clear from the first and second line that an $\omega^1{}_2$ proportional to θ^2 should do the job; similarly from the first and third line one sees that an $\omega^2{}_3$ proportional to θ^3 should work. It is then easy to find the relevant coefficient, as well as to find $\omega^2{}_3$:

$$\begin{aligned}
\omega^1{}_2 &= -\gamma' e^{-\beta}\theta^2 = -\gamma' e^{-\beta+\gamma} d\theta \ , &\quad (3.7.13a)\\
\omega^1{}_3 &= -\gamma' e^{-\beta}\theta^3 = -\gamma' e^{-\beta+\gamma}\sin\theta\, d\varphi \ , &\quad (3.7.13b)\\
\omega^2{}_3 &= -e^{-\gamma}\cot\theta\, \theta^3 = -\cos\theta\, d\varphi \ . &\quad (3.7.13c)
\end{aligned}$$

It is convenient to define *curvature two-forms*:

$$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\theta^c \wedge \theta^d \ . \qquad (3.7.14)$$

The *second Cartan structure equation* then reads

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b} \ . \qquad (3.7.15)$$

This identity is easily verified using (3.7.8):

$$\begin{aligned}
\Omega^a{}_b(X,Y) &= \frac{1}{2}R^a{}_{bcd}\underbrace{\theta^c\wedge\theta^d(X,Y)}_{=X^cY^d-X^dY^c}\\
&= R^a{}_{bcd}X^cY^d\\
&= \theta^a(\nabla_X\nabla_Y e_b-\nabla_Y\nabla_X e_b-\nabla_{[X,Y]}e_b)\\
&= \theta^a(\nabla_X(\omega^c{}_b(Y)e_c)-\nabla_Y(\omega^c{}_b(X)e_c)-\omega^c{}_b([X,Y])e_c)\\
&= \theta^a\Big(X(\omega^c{}_b(Y))e_c+\omega^c{}_b(Y)\nabla_X e_c\\
&\quad -Y(\omega^c{}_b(X))e_c-\omega^c{}_b(X)\nabla_Y e_c-\omega^c{}_b([X,Y])e_c\Big)\\
&= X(\omega^a{}_b(Y))+\omega^c{}_b(Y)\omega^a{}_c(X)\\
&\quad -Y(\omega^a{}_b(X))-\omega^c{}_b(X)\omega^a{}_c(Y)-\omega^a{}_b([X,Y])\\
&= \underbrace{X(\omega^a{}_b(Y))-Y(\omega^a{}_b(X))-\omega^a{}_b([X,Y])}_{=d\omega^a{}_b(X,Y)}\\
&\quad +\omega^a{}_c(X)\omega^c{}_b(Y)-\omega^a{}_c(Y)\omega^c{}_b(X)\\
&= (d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b)(X,Y)\ .
\end{aligned}$$

Equation (3.7.15) provides an efficient way of calculating the curvature tensor of any metric.

Example 3.7.1 continued: From (3.7.13) we find:

$$\begin{aligned}
\Omega^1{}_2 &= d\omega^1{}_2+\underbrace{\omega^1{}_1}_{=0}\wedge\omega^1{}_2+\omega^1{}_2\wedge\underbrace{\omega^2{}_2}_{=0}+\underbrace{\omega^1{}_3\wedge\omega^3{}_2}_{\sim\theta^3\wedge\theta^3=0}\\
&= -d(\gamma' e^{-\beta+\gamma}d\theta)\\
&= -(\gamma' e^{-\beta+\gamma})'dr\wedge d\theta\\
&= -(\gamma' e^{-\beta+\gamma})'e^{-\beta-\gamma}\theta^1\wedge\theta^2\\
&= \sum_{a<b}R^1{}_{2ab}\theta^a\wedge\theta^b\ ,
\end{aligned}$$

which shows that the only non-trivial coefficient (up to permutations) with the pair 12 in the first two slots is

$$R^1{}_{212}=-(\gamma' e^{-\beta+\gamma})'e^{-\beta-\gamma}\ . \tag{3.7.16}$$

A similar calculation, or arguing by symmetry, leads to

$$R^1{}_{313}=-(\gamma' e^{-\beta+\gamma})'e^{-\beta-\gamma}\ . \tag{3.7.17}$$

Finally,

$$\begin{aligned}
\Omega^2{}_3 &= d\omega^2{}_3+\omega^2{}_1\wedge\omega^1{}_3+\underbrace{\omega^2{}_2}_{=0}\wedge\omega^2{}_3+\omega^2{}_3\wedge\underbrace{\omega^3{}_3}_{=0}\\
&= -d(\cos\theta\, d\varphi)+(\gamma' e^{-\beta}\theta^2)\wedge(-\gamma' e^{-\beta}\theta^3)\\
&= (e^{-2\gamma}-(\gamma')^2e^{-2\beta})\theta^2\wedge\theta^3\ ,
\end{aligned}$$

yielding

$$R^2{}_{323}=e^{-2\gamma}-(\gamma')^2e^{-2\beta}\ . \tag{3.7.18}$$

The curvature scalar can easily be calculated now to be

$$\begin{aligned} R = R^{ij}{}_{ij} &= 2(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}) \\ &= -4(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} + 2(e^{-2\gamma} - (\gamma')^2 e^{-2\beta}) \,. \end{aligned} \tag{3.7.19}$$

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